Totally odd subdivisions and parity subdivisions: Structures and Coloring

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Abstract

A totally odd $H$-subdivision means a subdivision of a graph $H$ in which each edge of $H$ corresponds to a path of odd length. Thus this concept is a generalization of a subdivision of $H$.

In this paper, we give a structure theorem for graphs without a fixed graph $H$ as a totally odd subdivision. Namely, every graph with no totally odd $H$-subdivision has a tree-decomposition such that each piece is either

1. after deleting bounded number of vertices, an “almost” embedded graph into a bounded-genus surface, or
2. after deleting bounded number of vertices, a bipartite graph, or
3. after deleting bounded number of vertices, a graph with maximum degree at most $f(|H|)$ for some function $f$ of $|H|$ (or a $6|H|$-degenerate graph).

Moreover, we can obtain either a totally odd $K_k$-subdivision or such a tree-decomposition in polynomial time.

We note that for minor-free graphs, we just need the first structure [37], while for odd-minor-free graphs, we need the first and the third structures [17, 29]. So our result can be viewed as a combination of odd-minor-free graphs and subdivision-free graphs. The same conclusion of the structure theorem is true if we replace “totally odd” by “parity”. Hence this generalizes the structure theorem for subdivision-free graphs [17, 29].

We also consider coloring of graphs with no totally odd $K_k$-subdivision. We prove that any graph with no totally odd $K_k$-subdivision is $79k^2/4$-colorable. The bound on the chromatic number is essentially best possible since the correct order of the magnitude of the chromatic number even for graphs with no $K_k$-subdivision is $\Theta(k^2)$. Our result improves the bound given by Thomassen [44]. Furthermore, it also generalizes the result by Bollobás and Thomason, and Komlós and Szemeredi, [6, 30] for graphs without $K_k$-subdivisions.

Finally we consider coloring graphs with no totally odd $K_k$-subdivision in terms of an algorithmic view. Using our structure theorem, we give an approximation algorithm for coloring of a graph $G$ without a fixed graph $H$ as a totally odd subdivision, using $2\chi(G) + 6(|H| - 1)$ colors, where $\chi(G)$ is chromatic number of $G$. The same conclusion is true if we replace “totally odd” by “parity”.

We point out that it is Unique-Game hard to obtain an $O(k/\log^2 k)$-approximation algorithm for graph-coloring of graphs with maximum degree at most $k - 2$ [2], and hence it is also Unique-Game hard to obtain an $O(k/\log^2 k)$-approximation algorithm for graph-coloring of graphs even without a $K_k$-subdivision (i.e., without the parity constraint). Thus our additive error $\Theta(k)$ is most likely best possible up to constant.

Key Words: Totally odd subdivision, structure theorem, coloring, polynomial time

1 Introduction

1.1 Subdivision A graph $G$ contains a subdivision of a graph $H$ if $G$ contains a subgraph which is isomorphic to a graph that can be obtained from $H$ by subdividing some edges. Although the well-known Kuratowski’s theorem can be stated in terms of both a subdivision and a minor, the notions of a subdivision and a minor do not seem to be similar. For example, graphs without $K_5$-minors are already characterized in 1937 by Wagner [49], but graphs without $K_5$-subdivisions are still mysterious and perhaps it is out of reach to characterize graphs without $K_5$-subdivisions. Another example is that Robertson and Seymour [38] proved the famous Wagner’s conjecture which says that graphs are well-quasi-ordered by the minor relation. However, this is no longer true for the subdivision relation.
1.2 Totally odd subdivision A totally odd $H$-subdivision is a subdivision of $H$ where each subdivided edge has odd length. The study of totally odd $K_4$-subdivisions plays an important role in both graph theory and combinatorial optimization. In graph theory, the importance of such subdivisions arises from graph coloring. Toft [48] conjectured that any graph with no totally odd $K_4$-subdivision is 3-colorable. This is a generalization of Hajós’ conjecture for graphs without a $K_4$-subdivision, which was proved independently by Hadwiger [19] and Dirac [13]. It was finally settled by Zang [56] and Thomassen [46], independently. This conjecture also relates to the so-called “t-perfect graphs” and “strongly t-perfect graphs” from the theory of perfect graphs. For more details, we refer the reader to Schrijver’s book [41].

In combinatorial optimization, there is an interesting min-max relation in graphs with no totally odd $K_4$-subdivision. A graph $G$ is said to be $\alpha$-critical if the cardinality $\alpha(G)$ (size of largest stable set of $G$) increases with the removal of any edge. For an arbitrary graph $G$, denote by $\tilde{\rho}(G)$ the minimum cost of a family of vertices, edges and odd cycles covering $V(G)$, where the cost of a vertex or an edge is 1, the cost of an odd cycle $C$ is $(|C| - 1)/2$, and the cost of a family is the sum of the costs of its elements. Clearly $\alpha(G) \leq \tilde{\rho}(G)$. But if $G$ contains no totally odd $K_4$-subdivision, then we have $\alpha(G) = \tilde{\rho}(G)$ as a corollary of Sewell and Trotter’s theorem [42]. In fact, they gave a polynomial time algorithm to find a maximum stable set in a graph with no totally odd $K_4$-subdivision. This answered a conjecture of Chvátal [8]. For more consequences of this result, we refer the reader to Schrijver’s book [41].

1.3 Minor, Subdivision and Graph Coloring

Our paper is also motivated by two famous conjectures concerning the chromatic number and the minor and the subdivision order, namely, Hadwiger’s conjecture and Hajós’ conjecture.

Hadwiger’s Conjecture from 1943 suggests a far-reaching generalization of the Four Color Theorem and is considered to be one of the deepest open problems in graph theory. It states that every loopless graph without a $K_k$-minor is $(k - 1)$-colorable. In 1937, Wagner [49] proved that the case $k = 5$ of the conjecture is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [40] proved that the case $k = 6$ also follows from the Four Color Theorem. The cases $k \geq 7$ are open, and even for the case $k = 7$, partial result in [22] is best known. In general, the best known upper bound is $O(k^{\sqrt{\log k}})$ [31, 43].

Hajós proposed a stronger conjecture that for all $k \geq 1$, every graph without a subdivision of the complete graph on $k$ vertices is $(k - 1)$-colorable. He already considered the conjecture in the 1940’s in connection with attacks on the Four Color Conjecture (now Theorem). For $k \leq 4$, the conjecture is true, and for $k = 5, 6$, it still remains open. But for every $k \geq 7$, it was disproved by Catlin [7]. In fact, Erdös and Fajtlowicz [15] proved that the conjecture is false for almost all graphs, see also Bollobás and Catlin [5]. Recently, Thomassen [47] gave many families of graphs that are counterexamples to Hajós conjecture. In fact, the order of the correct magnitude for the chromatic number of graphs without a $K_k$-subdivision is $\Theta(k^2)$ [6, 30]. These two conjectures are so famous to attract so many outstanding researchers, as we saw here.

A general result by Thomassen [44] says that for every natural number $k$, there is a function $f(k)$ such that every $f(k)$-chromatic graph has a subdivision of $K_k$ such that each edge in $K_k$ corresponds to a path in the subdivision of any prescribed parity. A totally odd $K_k$-subdivision is also interesting in this connection, as pointed out by Thomassen [45].

But unfortunately, the bound of the chromatic number given in [44] is far from best possible, even for the totally odd $K_k$-subdivision case. In this paper, we shall give an essentially best possible value for the chromatic number as stated in the abstract. Let us observe that the result in [6, 30] says that there exists a constant $c$ such that every graph with minimum degree $ck^2$ contains a subdivision of $K_k$. This implies that the chromatic number for graphs without a subdivision of $K_k$ is at most $ck^2 - 1$. Indeed, as pointed out in [6, 30], this chromatic number is best possible (up to constants).

Our proof is based on induction on the number of vertices. For the induction purpose, we shall prove the following stronger statement.

Theorem 1.1. For any vertex set $Z$ in $G$ with $|Z| \leq k^2$, either $G$ has a totally odd $K_k$-subdivision or any precoloring of the subgraph of $G$ induced by $Z$ can be extended to a $79k^2/4$-coloring of $G$.

Let us point out that the proof of Theorem 1.1 implies that the same conclusion of Theorem 1.1 holds if we replace “totally” by “parity”, i.e, a parity subdivision of a graph $H$ means a subdivision of $H$, where length of each subdivided edge can be assigned to any prescribed parity between odd and even.

1.4 Structure theorem The theory of graph minors was developed by Robertson and Seymour in a series of 23 papers published over more than thirty years.
The aim of the series of papers is to prove a single result: the graph minor theorem, which says that in any infinite collection of finite graphs there is one that is a minor of another. As with other deep results in mathematics, the body of theory developed for the proof of the graph minor theorem has also found applications elsewhere, both within graph theory and computer science. Yet many of these applications rely not only on the general techniques developed by Robertson and Seymour to handle graph minors, but also on one particular auxiliary result which is central to the proof of the graph minor theorem: a result which approximately describes the structure of all graphs which do not contain some fixed graph $H$ as a minor, see [37]. At a high level, the theorem says that every such a graph has a tree-decomposition such that each piece is

after deleting bounded number of vertices, an “almost” embedded graph into a bounded-genus surface.

Recently, similar structure results are obtained for graphs with broader family of graphs. In [10], Demaine, Hajiaghayi and Kawarabayashi obtained a structure theorem for odd-minor-free graphs. Namely, every graph with no odd $H$-minor has a tree-decomposition such that each piece is either

1. after deleting bounded number of vertices, an “almost” embedded graph into a bounded-genus surface, or

2. after deleting bounded number of vertices, a bipartite graph.

In another direction, Grohe and Marx [17], independently Kawarabayashi and Kobayashi [29] obtained a structure theorem for graphs without a fixed graph $H$ as a subdivision. Namely, every graph with no $H$-subdivision has a tree-decomposition such that each piece is either

1. after deleting bounded number of vertices, an “almost” embedded graph into a bounded-genus surface, or

2. after deleting bounded number of vertices, a graph with maximum degree at most $f(|H|)$ for some function $f$ of $|H|$ [17] (or a $6|H|$-degenerate graph [29] (a graph $G$ is $d$-degenerate if each induced subgraph of $G$ has a vertex of degree at most $d$)).

In this paper, we give a structure theorem for graphs without a fixed graph $H$ as a totally odd subdivision. Namely, every graph with no totally odd $H$-subdivision has a tree-decomposition such that each piece is either

1. after deleting bounded number of vertices, an “almost” embedded graph into a bounded-genus surface, or

2. after deleting bounded number of vertices, a bipartite graph, or

3. after deleting bounded number of vertices, a graph with maximum degree at most $f(|H|)$ for some function $f$ of $|H|$ (or a $6|H|$-degenerate graph).

Moreover, we can obtain either a totally odd $K_k$-subdivision or such a tree-decomposition in polynomial time. The same conclusion of the structure theorem is also true if we replace “totally odd” by “parity”. Hence this generalizes the structure theorem for subdivision-free graphs [17, 29].

We note that for minor-free graphs, we just need the first structure [37], while for odd-minor-free graphs, we need the first two structures [10]. For subdivision-free graphs, we need the first and the third structures [17, 29]. So our result can be viewed as a combination of odd-minor-free graphs and subdivision-free graphs.

1.5 Algorithmic Results for totally odd subdivisions and immersions We now consider the algorithmic question about coloring graphs without a fixed graph as a totally odd subdivision. Theorem 1.1 already tells us the upper bound of the chromatic number for such a graph, however we can do it better in terms of an algorithmic view. Let us observe that coloring graphs without a fixed graph as a minor or as an odd minor, in terms of an algorithmic view is already studied in [10, 24, 25, 26]. In this paper, we are most interested in an additive approximation algorithm, as Kawarabayashi and Kobayashi gave the corresponding additive approximation algorithm for coloring of graphs without a fixed graph as a subdivision [29].

**Theorem 1.2.** Let $G$ be a graph with no $K_k$-subdivision. There is a polynomial time algorithm to give a coloring of $G$ using at most $\chi(G) + 2.5(k + 1)$ colors.

In fact, the algorithm is stronger in the following sense: given a graph $G$ and fixed constant $k$, the algorithm either outputs a $K_k$-subdivision or gives a coloring of $G$ using at most $\chi(G) + 2.5(k + 1)$ colors.

In this paper, using our structure theorem, we give a polynomial time algorithm to color graphs with no totally odd $K_k$-subdivision.

**Theorem 1.3.** Let $G$ be a graph with no totally odd $K_k$-subdivision. Let $S \subseteq V(G)$ with $|S| \leq 6(k - 1)$. Given a precoloring of $S$, there is a polynomial time
algorithm to give a coloring of $G$ that extends the
precoloring of $S$, using at most $2\chi(G) + 6(k - 1)$ colors.

In fact, our algorithm is stronger in the following
sense; given a graph $G$, a precoloring of a vertex set $S \subseteq V(G)$ with $|S| \leq 6(k - 1)$ and fixed constant $k$, the
algorithm either outputs a totally odd $K_k$-subdivision or
gives a coloring of $G$ that extends the precoloring of $S$,
using at most $2\chi(G) + 6(k - 1)$ colors.

The same is true if we replace "totally odd $K_k$-
subdivision" by "parity subdivision".

An immersion of $H$ in $G$ is a map $\eta$ such that

- $\eta(v) \in V(G)$ for each $v \in V(H)$
- $\eta(u) \neq \eta(v)$ for distinct $u, v \in V(H)$
- for each edge $e = uv$ of $H$, $\eta(e)$ is a path of $G$ from
  $\eta(u)$ to $\eta(v)$
- if $e, f \in E(H)$ are distinct, then $\eta(e), \eta(f)$ have no
  edges in common, although they may share vertices

In such a case, we sometimes say that $G$ contains
an $H$-immersion. A totally odd $K_1$-immersion is an
immersion of $K_1$ where each subdivided edge has odd
length

Theorem 1.3 gives the following direct corollary.

**Theorem 1.4.** Let $G$ be a graph with no totally odd
$K_k$-immersion. There is a polynomial time algorithm
to give a coloring of $G$ using at most $2\chi(G) + 6(k - 1)$
colors.

In fact, our algorithm is stronger in the following
sense; given a graph $G$ and fixed constant $k$, the
algorithm either outputs a totally odd $K_k$-immersion or
gives a coloring of $G$, using at most $2\chi(G) + 6(k - 1)$
colors.

The same is true if we replace "totally odd $K_k$-
immersion" by "parity immersion" (here, parity immer-
sion). We can mention some hardness result that is related
to Theorems 1.2, 1.3 and 1.4. It is Unique-Game hard
to obtain an $O(k/\log^2 k)$-approximation algorithm for
graph-coloring of a graph $G$ with maximum degree at
most $k - 2$ [2]. Clearly $G$ does not contain $K_k$ as a
subdivision nor as an immersion (i.e., without the parity
constraint), and hence it is also Unique-Game hard to
obtain an $O(k/\log^2 k)$-approximation algorithm for
graph-coloring of graphs even without a $K_k$-subdivision
or without a $K_k$-immersion. Therefore it really makes
sense to consider an approximation algorithm with
additive factor for graph-coloring of graphs with no
totally odd $K_k$-subdivision or with no totally odd $K_k$-
immersion. Thus our additive error $\Theta(k)$ is most likely
best possible up to constant.

We also note that Theorem 1.1 tells us that any
graph with no totally odd $K_k$-subdivision is $O(k^2)$-
colorable, and as mentioned above, there may be such a
graph that needs as many as $\Theta(k^2)$ colors. But on the
other hand, Theorem 1.3 tells us that if such a graph
is $o(k^2)$-colorable, our algorithm gives rise to an $o(k^2)$-
coloring too (rather than $k^2$-coloring as in Theorem 1.1).

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

If $V(G) = Z$, there is nothing to prove. Let $G$ be
a minimal counterexample with $|G|$ minimum. That
is, $G$ has no totally odd $K_k$-subdivisions and $G$ is not
$79k^2/4$-colorable, but any proper subgraph of $G$ has a
$79k^2/4$-coloring. Suppose $G - Z$ has a vertex $v$ of degree
at most $79k^2/4 - 1$. Then by the minimality, $G - v$ has
a $79k^2/4$-coloring. Clearly this coloring can be extended
to $G$, a contradiction. So we may assume that every
vertex $v \in V(G - Z)$ has degree at least $79k^2/4$.

Let $A$ and $B$ be induced subgraphs of $G$ such
that $G = A \cup B$. We say that the pair $(A, B)$ is a
separation of $G$. The order of this separation is equal to
$|V(A \cap B)|$. Let $Z \subseteq V(G)$ be a vertex set. We say that
the separation $(A, B)$ of $G$ is $Z$-essential if $(A, B)$ is a
separation of $G$ such that both $A - B - Z$ and $B - A - Z$
are nonempty. We shall prove the following.

(1) There is no $Z$-essential separation $(A, B)$ of order at
most $k^2/2$ in $G$.

Suppose that there is such a separation $(A, B)$ of order
at most $k^2/2$. We assume that $(A, B)$ is a minimal
$Z$-essential separation, and we let $S = A \cap B$. Since $|S| \leq
k^2/2$ and $|Z| \leq k^2$, it follows that either $|S \cup (A \cap Z)| \leq
k^2$ or $|S \cup (B \cap Z)| \leq k^2$, say $|S \cup (A \cap Z)| \leq k^2$. Since
$(A, B)$ is a $Z$-essential separation, the subgraph on $B \cup Z$
is smaller than $G$. Thus by the minimality of $G$, we get
a $79k^2/4$-coloring of the subgraph of $G$ induced by $B \cup Z$
with $Z$ precolored.

Again, by the minimality of $G$, we get a $79k^2/4$-
coloring of $B$ with $Z' = S \cup (A \cap Z)$ precolored, where
the precoloring of vertices in $S$ comes from the coloring of
$B \cup Z$. Recall that $|Z'| \leq k^2$ and the subgraph induced
on $A$ is smaller than $G$. Finally, the combination of the
obtained colorings of $B$ and $A$ yields a $79k^2/4$-coloring
of $G$, a contradiction.

We choose a spanning bipartite graph $H$ of $G - Z$ such
that the minimum degree of $H$ is as large as possible.
The old theorem of Erdős says that every graph of the
minimum degree at least 2l has a spanning bipartite graph of the minimum degree at least l. Since \( G - Z \) has minimum degree at least \( 79k^2/4 - k^2 = 75k^2/4 \), so \( H \) has minimum degree at least \( 75k^2/8 \). We prove the following

(2) \( H \) has a \( 3k^2/4 \)-linked subgraph \( L \).

To prove (2), we shall need the following result in [4]. A graph \( L \) is said to be \( k \)-linked if it has at least \( 2k \) vertices and for any ordered \( k \)-tuples \((s_1, \ldots, s_k) \) and \((t_1, \ldots, t_k) \) of \( 2k \) distinct vertices of \( L \), there exist pairwise vertex-disjoint paths \( P_1, \ldots, P_k \) such that for \( i = 1, \ldots, k \), the path \( P_i \) connects \( s_i \) and \( t_i \).

(2.1) Let \( G \) be a graph and \( l \) an integer such that

(a) \( |V(G)| \geq 5/2l \) and

(b) \(|E(G)| \geq 25/4l|V(G)| - 25l^2/2.\)

Then \( G \) contains an \( l \)-linked subgraph \( R \). In particular, \( R \) is \( 2l \)-connected with at least \( 5l|R| \) edges.

(2.1) immediately implies (2) since the minimum degree of \( H \) is at least \( 75k^2/8 \).

Let \( L \) be a \( 3k^2/4 \)-linked bipartite subgraph. We say that \( P \) is a parity breaking path for \( L \) if \( P \) is disjoint from \( L \) except for its endpoints, and \( L \) together with \( P \) has an odd cycle. This parity breaking path may be just a single edge.

We need the odd \( S \)-path theorem which is a generalization of the well-known Mader’s \( S \)-path theorem. We shall use the following recent result in [16].

**Theorem 2.1.** For any set \( S \) of vertices of a graph \( G \), and for any fixed \( k \), either

1. there are \( k \) disjoint odd \( S \) paths, i.e., \( k \) disjoint paths each of which has an odd number of edges and both its endpoints in \( S \), or

2. there is a vertex set \( X \) of order at most \( 2k - 2 \) such that \( G - X \) contains no such paths.

In our proof of Theorem 1.1, we do not need to consider an algorithmic aspect of Theorem 2.1, however, for our future purpose, let us observe that the proof given in [16] actually implies that there is a polynomial time algorithm to give one of the conclusions in Theorem 2.1. In fact, the algorithm reduces the problem to the maximum matching problem. So we can obtain an \( O(mn) \) time algorithm for Theorem 2.1.

Since \( L \) is \( 3k^2/4 \)-linked and hence \( L \) is \( 3k^2/2 \)-connected (see [11] for example), one of the partite sets of \( L \) has at least \( 3k^2/2 \) vertices.

(3) There are at least \( 1/2k^2 \) disjoint parity breaking paths for \( L \).

Take one partite set \( S \) of \( L \) with at least \( 3k^2/2 \) vertices. We shall apply Theorem 2.1 to \( G \) and \( S \). If there are at least \( k^2/2 \) vertex disjoint odd \( S \) paths in \( G \), we can clearly find \( k^2/2 \) vertex-disjoint parity breaking paths for \( L \). Otherwise, by Theorem 2.1, there is a vertex set \( X \) of order at most \( k^2 - 2 \) such that \( G - X \) has no such paths. It follows from Theorem 2.1 that \( G - X \) can be written as \( L' \cup W_1 \cup W_2 \cup \cdots \cup W_m \) for some integer \( m \) such that \( L' \) is an induced bipartite graph containing \( L - X \) (because \( L \) is \( 3k^2/2 \)-connected), and each \( W_i \) is a block. In addition, each \( W_i \) contains at most one vertex \( v_i \) which is also contained in \( L' \), such that \( W_i - v_i \) is disjoint from \( L' \). We claim that \( W_i - v_i \) consists of vertices in \( Z \) \((1 \leq i \leq m) \). Otherwise, there would be a separation \((A, B)\) of order at most \(|X| + 1 \leq k^2 \) in \( G \) such that both \( A - B - Z \) and \( B - A - Z \) are nonempty, a contradiction to (1).

Because \( W_1 - Z - v_1 = \emptyset \), we need at most \( k^2 \) colors for \( G - X - L' \). We only need 2 colors for \( L', \) and at most \( k^2 - 2 \) colors for \( X \), since \(|X| \leq k^2 - 2 \). Clearly these colors are enough to color \( G \), and we only need at most \( 2k^2 \) colors for the coloring of \( G \), a contradiction. This completes the proof of (3). ■

We now have a \( 3k^2/4 \)-linked graph with \( 1/2k^2 \) parity breaking paths \( P_1, P_2, \ldots, P_{1/2k^2} \). Let \( p_i, p'_i \) be the two endpoints of the parity breaking path \( P_i \) for \( i = 1, \ldots, 1/2k^2 \). Let \( W \) be their union. So \( W \) consists of \( k^2 \) vertices of \( L \). Let us assume that \( L \) consists of the bipartition \((L_1, L_2)\). Since the minimum degree of \( L \) is at least \( 3k^2/2 \), so \(|L_1|, |L_2| \geq 3k^2/2 \). Take a vertex set \( K_1 \) in \( L_1 - W \) and a vertex set \( K_2 \) in \( L_2 - W \) such that \(|K_1| = |K_2| = k/2 \). Let \( K_1 = \{k_1, \ldots, k_{k/2}\} \) and \( K_2 = \{k_{k/2+1}, \ldots, k_{k}\} \). We shall now construct a totally odd \( K_k \)-subdivision such that branch vertices are \( K_1 \cup K_2 \). Since the minimum degree of \( L \) is at least \( 3k^2/2 \), there are \( 2k^2/2 - k \) vertices \( \{k_{i,1}, k_{i,2}, k_{i,3}, \ldots, k_{k,1}, k_{k,2}, k_{k,3}, \ldots, k_{k,k-1}\} \) in \( L_2 - W - K_2 \) such that for all \( j \) \((2 \leq j \leq k - 1)\), \( k_{i,j} \) is adjacent to \( k_i \) for each \( i \) \((1 \leq i \leq k/2)\). Similarly, there are \( 2k^2/2 - k \) vertices \( \{k_{k/2+1,2}, k_{k/2+1,3}, \ldots, k_{k/2+1,k-1}, k_{k/2+2,2}, \ldots, k_{k/2+2,k-1}, k_{k/2+3,2}, \ldots, k_{k,k-1}\} \) in \( L_1 - W - K_1 \) such that for all \( j \) \((2 \leq j \leq k - 1)\), \( k_{i,j} \) is adjacent to \( k_i \) for each \( i \) \((k/2 + 1 \leq i \leq k)\).

Since \( L \) is \( 3k^2/4 \)-linked, by choosing the appropriate labeling of \( k_i, k_{i,j}, p_i, p'_i \) for \( i, j, l \), we can obtain a totally odd \( K_k \)-subdivision with branch vertices \( K_1 \cup K_2 \) as follows.
1. a $K_{k/2,k/2}$-subdivision in $L$ such that branch vertices of this subdivision are $K_1 \cup K_2$, and

2. two disjoint totally odd $K_{k/2}$ subdivisions such that branch vertices of these two totally odd $K_{k/2}$-subdivisions are $K_1, K_2$ respectively, and each odd path corresponding to the edge of the totally odd subdivisions uses exactly one parity breaking path.

Let us observe that in order to obtain such a configuration, we need at most $3k^2/4$ disjoint paths with endpoints $k_1, k_{i,j}, p_1, p_1'$ in $L$. As remarked above, these disjoint paths can be found because $L$ is $3k^2/4$-linked. This gives rise to a totally odd $K_k$-subdivision, a contradiction. This completes the proof.\]

3 Preliminaries for the rest of the paper

We now give a proof for our structure theorem. In order to do that, we need to introduce some notations. In this paper, $n$ and $m$ always mean the number of vertices of a given graph and the number of edges of a given graph, respectively.

We now look at definitions of the tree-width and the clique model.

**Tree-width** Let $G$ be a graph, $T$ a tree and let $V = \{V_t \subseteq V(G) \mid t \in V(T)\}$ be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices $t$ of $T$. The pair $(T, V)$ is called a tree-decomposition of $G$ if it satisfies the following three conditions:

- $V(G) = \bigcup_{t \in T} V_t$,
- for every edge $e \in E(G)$ there exists a $t \in T$ such that both ends of $e$ lie in $V_t$,
- if $t, t', t'' \in V(T)$ and $t'$ lies on the path of $T$ between $t$ and $t''$, then $V_t \cap V_{t'} \subseteq V_{t''}$.

The width of $(T, V)$ is the number $\max\{|V_t| - 1 \mid t \in T\}$ and the tree-width $tw(G)$ of $G$ is the minimum width of any tree-decomposition of $G$. Sometime, we refer $V_t$ to as a bag.

Robertson and Seymour developed the first polynomial time algorithm for constructing a tree decomposition of a graph of bounded width [34], and eventually came up with an algorithm which runs in $O(n^2)$ time, for this problem. Bodlaender [3] developed a linear time algorithm.

**Theorem 3.1.** For an integer $w$, there exists a $(wO(w))nO(1)$ time algorithm that, given a graph $G$, either finds a tree-decomposition of $G$ of width $w$ or concludes that the tree-width of $G$ is more than $w$. Furthermore, if $w$ is fixed, there exists an $O(n)$ time algorithm to construct one of them.

We can apply dynamic programming to solve the graph coloring problem on graphs of bounded tree-width, in the same way that we apply it to trees (see e.g. [1]), provided that we are given a bounded width tree decomposition. Thus Theorem 3.1 together with [1] implies the following.

**Theorem 3.2.** For integers $w$ and $k$, there exists a $(wO(kw))nO(1)$ time algorithm to determine $\chi(G)$ in a graph $G$ of tree-width $w$. Moreover, if $w$ and $k$ are fixed, there exists an $O(n)$ time algorithm to color $G$ using $\chi(G)$ colors.

**Clique model** For an integer $p$, $K_p$ is the complete graph with $p$ vertices. A graph $G$ contains a $K_p$-model if there exists a function $\sigma$ with domain $V(K_p) \cup E(K_p)$ such that

1. for each vertex $v \in V(K_p)$, $\sigma(v)$ is a connected subgraph of $G$, and the subgraphs $\sigma(v)$ ($v \in V(K_p)$) are pairwise vertex-disjoint, and

2. for each edge $e = uv \in E(K_p)$, $\sigma(e)$ is an edge $f \in E(G)$, such that $f$ is incident in $G$ with a vertex in $\sigma(u)$ and with a vertex in $\sigma(v)$.

Thus $G$ contains a $K_p$-minor if and only if $G$ contains a $K_p$-model. We call the subgraph $\sigma(v)$ ($v \in V(K_p)$) the node of the $K_p$-model. The image of $\sigma$, which is a subgraph of $G$, is called the $K_p$-model.

We say that a $K_p$-model is even if the union of the nodes of the $K_p$-model consists of a bipartite graph. We also say that a $K_p$-model is odd if for each cycle $C$ in the union of the nodes of the $K_p$-model, the number of edges in $C$ that belong to nodes of the $K_p$-model is even.

We need the following theorem.

**Theorem 3.3.** (Robertson and Seymour [34]) Let $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ be the terminals in a given $G$. If there is a clique model of order at least $3k$ in $G$, and there is no separation $(A, B)$ of order at most $2k - 1$ in $G$ such that $A$ contains all the terminals and $B - A$ contains at least one node of the clique model, then there are vertex-disjoint paths $P_i$ with two ends in $s_i, t_i$ for $i = 1, \ldots, k$.

The next theorem, which is the parity version of Theorem 3.3, will be used in our proof. For the proof, see [28].

**Theorem 3.4.** Let $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ be the terminals in a given $G$. If $G$ has an odd-$K_{2k}$-model, and there is no separation $(A, B)$ of order at most $2k - 1$ in $G$ such that $A$ contains all the terminals and $B - A$ contains at least one node of the odd clique model. Then
G has k vertex disjoint paths $P_1, \ldots, P_k$ such that $P_i$ joins $s_i$ and $t_i$, for $1 \leq i \leq k$, and we can specify any parity (i.e., even or odd) for $P_i$.

4 Large clique models I: Even clique models with many odd paths
Suppose that a $K_p$-model $L$ in $G$ is given. Let $v \in V(K_p)$. A center for $v$ is a vertex $x \in V(\sigma(v))$ such that for each component $H$ of $\sigma(v) - t$, the number of edges $e \in E(K_p)$ such that $\sigma(e)$ is incident in $G$ with a vertex of $H$ is at most half the number of edges in $K_p$ incident with $v$. It is not hard to see that every node $\sigma(v)$ has a center (perhaps more than one). Thus we assume that for each node, one of its centers has been selected, and we often speak of the center of a node without further explanation.

Geelen et al. [16] proved the following result.

**Theorem 4.1.** Suppose $G$ has an even clique model $L$ of order at least 16k. Then either

1. $G$ has 2k disjoint nodes $N_1, \ldots, N_{2k}$ of $L$ (that consist of disjoint trees), their centers $c_1, \ldots, c_{2k}$, and $k$ disjoint odd paths $P_1, \ldots, P_k$, such that for $i = 1, \ldots, k$, the endpoints of $P_i$ are $c_{2i-1}, c_{2i}$ in $P_i$, and $P_i$ does not intersect any node $N_j$ with $j \neq 2i - 1, 2i$ (and moreover, $N_{2i-1} \cup N_{2i} \cup P_i$ for $i = 1, \ldots, k$ gives rise to an odd $K_k$-model), or

2. $G$ has a vertex set $X$ with $|X| < 8k$ such that the block $F$ intersecting at least 8k disjoint nodes of $L$ in $G - X$ is bipartite.

The following is our main result in this section. This follows from Kawarabayashi and Song [23] and Geelen et al. [16], but for the completeness, we give a sketch of proof.

**Theorem 4.2.** Suppose $G$ has a $K_{16k, \sqrt{\log k}}$-model $L$. Then $G$ has an even $K_{16k}$-model. Consequently, either

1. $G$ has 2k disjoint trees $N_1, \ldots, N_{2k}$ that consists of an even $K_{2k}$-model, their centers $c_1, \ldots, c_{2k}$, and $k$ disjoint odd paths $P_1, \ldots, P_k$, such that for $i = 1, \ldots, k$, the endpoints of $P_i$ are $c_{2i-1}, c_{2i}$ in $P_i$, and $P_i$ does not intersect any node $N_j$ with $j \neq 2i - 1, 2i$ (and moreover, $N_{2i-1} \cup N_{2i} \cup P_i$ for $i = 1, \ldots, k$ gives rise to an odd $K_k$-model), or

2. $G$ has a vertex set $X$ with $|X| < 8k$ such that the block $F$ intersecting at least 8k disjoint nodes of $L$ in $G - X$ is bipartite.

**Sketch of Proof.** Since $G$ has a $K_{16k, \sqrt{\log k}}$-model, there are $l = 32k \sqrt{\log k}$ disjoint trees $T_1, T_2, \ldots, T_l$ such that for any $i$ and $j$ with $i \neq j$, there is an edge between $T_i$ and $T_j$. Choose a two coloring of all the trees $T_1, T_2, \ldots, T_l$ such that the number of bichromatic edges between two of the trees is as many as possible. Clearly, for each tree $T_i$, there are at least $l/2$ bichromatic edges leaving from $T_i$, otherwise, swapping the coloring of $T_i$ would give rise to the bigger number of bichromatic edges leaving from $T_i$. Therefore, the graph induced by the trees $T_1, T_2, \ldots, T_l$ and bichromatic edges between two of the trees is bipartite. So we have a model $H$ which is a a spanning bipartite subgraph of the $K_{32k \sqrt{\log k}}$-model such that the minimum degree of nodes of $H$ is at least $8k \sqrt{\log k}$. Moreover, each node of $H$ is a node of the $K_{32k \sqrt{\log k}}$-model $L$. Since the minimum degree at least $k/4 \sqrt{\log k}/4$ guarantees a $K_k$-model by [31, 43], $H$ has an even $K_{16k}$-model $L'$ such that each node of $L'$ consists of node(s) of $H$. By Theorem 4.1 applied to $L = L'$, Theorem 4.2 follows.

As pointed out in the previous section, the above proof gives rise to a polynomial time algorithm for Theorem 4.2, provided that a $K_{16k, \sqrt{\log k}}$-model is given. Translating the $K_{16k, \sqrt{\log k}}$-model to the even $K_{16k}$-model can be clearly done in linear time. Then by the algorithm of Theorem 2.1, as in the proof by Geelen et al. [16], we are able to obtain one of the conclusions in Theorem 4.2 in polynomial time.

5 Large clique model II: Using spiders
Let $A$ and $B$ be two disjoint sets of vertices in a given graph $G$. A $d$-spider with branch $v$ is a set of $d$ vertex-disjoint paths connecting $v \in A$ with $B$. The following is proved in [39], Theorem (7.2). A shorter proof is also given by Dániel Marx (private communication).

**Theorem 5.1.** There is a function $f(k,d) = 2^{O(kd)}$ such that for every graph $G$ and disjoint vertex sets $A,B \in V(G)$ either

1. there are $k$ vertex-disjoint $d$-spiders, or

2. there is a set $D$ of at most $f(k,d)$ vertices that intersects every $d$-spider.

Furthermore, given a graph $G$ and $A,B$, and given integers $k,d$, there is a polynomial time algorithm to construct one of the above conclusions.

The objective of this section is to consider the case when $G$ has a large even clique model with many odd paths joining centers, and there are large number of “spiders” with respect to the even clique model. In

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This slides contain a shorter proof, see http://www.cs.bme.hu/~dmarx/papers/marx-mds-separators-slides.pdf
this case, we shall show that there is a totally odd \(K_k\)-subdivision.

Let \( p \geq 30k^2 \) be an integer. We assume that an even \( K_p\)-model \( L \) in \( G \) is given.

We now label each vertex of \( K_p \) as \( 1, \ldots, p \). For each \( \sigma(i) \) \( (1 \leq i \leq p) \), we pick up the center \( c_i \). Let \( C = \{ c_1, \ldots, c_p \} \).

We now add the vertices \( c'_1, \ldots, c'_p \) to \( G \) such that \( c'_i c_i \) is an edge for \( i = 1, \ldots, p \). Let \( G' \) be the resulting graph. So \( c'_i \) is of degree one in \( G' \). Let \( S = \{ c'_1, \ldots, c'_p \} \).

We can easily extend the definition of the node \( \sigma(i) \) in \( G \) to that in \( G' \) to include each vertex of \( S \) in some node \( \sigma(i) \). We note that the model obtained from \( K_p \) as above is still an even clique model. We now suppose the following.

**Hypothesis**

There are \( 13k^2 \) vertex-disjoint \( S \)-odd paths \( P' = P'_1, \ldots, P'_{13k^2} \) such that the endpoints of \( P'_1 \) are \( c'_{2i-1} c'_{2i} \) and in addition \( P'_1 \) does not intersect any vertex in \( \sigma(j) \) with \( j \neq 2i - 1, 2i \), for \( i = 1, \ldots, 13k^2 \).

Hence \( \sigma(2i - 1) \cup \sigma(2i) \cup P'_i \) for \( i = 1, \ldots, 13k^2 \) gives rise to an odd \( K_{13k^2} \)-model.

Our purpose of this section is to show that if there are many vertex-disjoint \( d \)-spiders with \( A = V(G') - S \) and \( B = S \) in \( G' \) and \( d \geq 1.5k \), then we get a totally odd \( K_k \)-subdivision in \( G \). To do so, we need a few lemmas.

**Lemma 5.2.** Assuming Hypothesis, if there are \( 2k \) vertex-disjoint \( 1.5k \)-spiders with \( A = V(G') - S \) and \( B = S \) in \( G' \), at most \( 3k^2 \) paths in \( P' \) intersect some vertex contained in these spiders.

**Proof.** Let \( Q \) be the vertices of degree one in the \( 2k \) vertex-disjoint \( 1.5k \)-spiders. Thus \( Q \) consists of exactly \( 3k^2 \) vertices in \( S' \).

Denote branches \( S'' \) by the vertices of degree \( 1.5k \) in the the \( 2k \) vertex-disjoint \( 1.5k \)-spiders. Thus there are exactly \( 2k \) branches. Therefore the \( 2k \) vertex-disjoint \( 1.5k \)-spiders consist of \( 2k \) pairs of \( 1.5k \) disjoint paths \( P_{1,2}, \ldots, P_{1,5k,4} \) with the branch \( s_1 \).

We choose the paths \( P = \{ P_{1,1}, \ldots, P_{1,5k,1}, \ldots, P_{1,2}, \ldots, P_{1,5k,2k} \} \) such that

1) the number of paths in \( P' \) that hit a vertex in \( P \) is as small as possible, and subject to that,

2) the number of components in \( E(P) \cap E(P') \) is as small as possible

If an odd path \( P' \) in \( P' \) with two endvertices \( c'_{2i-1}, c'_{2i} \) (note that by our choice, the endpoints of \( P' \) are centers) intersects at least two paths in \( P \), let \( W \) and \( W' \) be the paths in \( P \) whose intersections are as close as possible (on \( P' \)) to \( c_{2i-1} \) and \( c_{2i} \), respectively. If \( W = W' \), suppose that the intersection \( u \) of \( W \) with \( P \) nearest \( c'_{2i} \) (say) comes before the intersection nearest \( c'_{2i-1} \). Let us replace the segment of \( W \) from \( u \) to \( P' \) with the segment of \( P' \) from \( u \) to \( c_{2i} \). This yields a contradiction to either 1) or 2), and shows that \( W \neq W' \). The same argument applied to \( W \) and \( W' \) shows that we may assume that \( W \) ends at \( c_{2i-1} \) and that \( W' \) ends at \( c_{2i} \).

Moreover, the same argument applies if an odd path \( P' \) in \( P' \) intersects exactly one path in \( P \).

This implies that if an odd path \( P' \) in \( P' \) with two endvertices \( c'_{2i-1}, c'_{2i} \) intersects a path in \( P' \), at least one path in \( P \) ends at either \( c_{2i-1} \) or \( c_{2i} \). Since \( |P'| = 3k^2 \), thus the result holds.

We now show the following main lemma.

**Lemma 5.3.** Assuming Hypothesis and Lemma 5.2, if there are \( 2k \) vertex-disjoint \( 1.5k \)-spiders with \( A = V(G) - S \) and \( B = S \) in \( G' \), then \( G \) has a totally odd \( K_k \)-subdivision.

**Proof.** We follow the notations given in the proof of Lemma 5.2.

We choose \( P \) so that the paths \( \{ P_{1,1}, \ldots, P_{1,5k,1}, \ldots, P_{1,2}, \ldots, P_{1,5k,2k} \} \) go through at most \( 3k^2 \) nodes \( \sigma(j) \). Such a choice is possible because there are exactly \( 3k^2 \) internally disjoint paths in \( P \) and some of the paths in \( P \) hitting the node \( \sigma(j) \) can end with the vertex \( \sigma(j) \cap S \) (i.e., \( c'_j \)). We relabel the nodes \( \sigma(j) \) such that the node \( \sigma(j) \) does not hit any of the paths in \( P \) for \( j = 6k^2 + 1, \ldots, p \). Each node \( \sigma(j) \) with \( j \geq 6k^2 + 1 \) has a neighbor in the node \( \sigma(j') \) with \( j' \leq 6k^2 \), where the node \( \sigma(j') \) intersects a path in \( P \).

By rerouting, we mean a path \( P \) that is obtained from a subpath \( P_1 \) of a path \( P' \) in \( P \) between \( s \in S'' \) and \( q \), together with a path \( P_2 \) such that:

1. \( P_2 \) starts at the vertex \( \sigma(j) \cap S \) (i.e., \( c'_j \)) with \( j \geq 6k^2 + 1 \) and ends at \( q \).
2. \( P_2 \) does not intersect any path in \( P \), except for \( P' \) (in fact \( P_1 \)), and
3. \( P_2 \) does not intersect any node \( \sigma(i) \) except for the node \( \sigma(j) \) and the one containing \( q \).

Thus the path \( P = P_1 \cup P_2 \) starts at the branch \( s \in S'' \) through \( q \) and ends at the vertex \( \sigma(j) \cap S \) (i.e., \( c'_j \)), and moreover \( P \) intersects exactly one path \( P' \) in \( P \).

The idea of the rerouting is the following: For fixed index \( j_1 \) with \( 1 \leq j_1 \leq 6k^2 \) such that the node \( \sigma(j_1) \) hits at least one path in \( P \), we have the following fact:
[\textsuperscript{[\ast]}] for any \( j_2 \geq 6k^2 + 1 \), some path in \( \mathcal{P} \) can be rerouted at the node \( \sigma(j_1) \) so that the resulting rerouted path starts at a branch in \( S' \) and ends at the vertex \( \sigma(j_2) \cap S \) (i.e., \( c'_j \)).

We now modify the paths in \( \mathcal{P} \) by sequences of reroutings so that:

1. each rerouted path ends at the vertex \( \sigma(j') \cap S \) (i.e., \( c'_j \)) with \( j' \geq 6k^2 + 1 \) and
2. \( \sigma(j') \) hits at most one path out of the resulting rerouted paths and if it hits the rerouted path \( P \), \( P \) must end at the vertex \( \sigma(j') \cap S \) (i.e., \( c'_j \)) for \( j' \geq 6k^2 + 1 \).

Our rerouting algorithm proceeds as follows: Initially, we set up \( \mathcal{P}' \) empty.

We pick up indices \( j_1, j_2 \) with \( j_1 \leq 6k^2 \) and \( j_2 \geq 6k^2 + 1 \), and a path \( P \in \mathcal{P} \) hitting the node \( \sigma(j_1) \), such that, letting \( P' \) be the subpath of \( P \) between its branch and the node \( \sigma(j_1) \),

1. it is not possible to reroute \( P \) at any other node \( \sigma(j'_1) \) (with \( j_1 \neq j'_1 \)) that the subpath \( P' \) goes through (since some other path in \( \mathcal{P} \) is already rerouted at the node \( \sigma(j'_1) \)); it is not possible to reroute the subpath \( P' \) at the node \( \sigma(j'_1) \) because the rerouted path at the node \( \sigma(j'_1) \) “blocks” the rerouting of the subpath \( P' \), and
2. we can reroute \( P \) at the node \( \sigma(j_1) \) and the resulting rerouted path of \( P \) ends at the vertex \( \sigma(j_2) \cap S \) (i.e., \( c'_j \)).

We discard \( P \) from \( \mathcal{P} \). Then add the rerouting rerouted path to the set \( \mathcal{P}' \). We do this process as much as possible (to maximize \(| \mathcal{P}' | \)).

We claim \(| \mathcal{P}' | \geq 2k^2 \). Suppose there is a path \( P \in \mathcal{P} \) that is not rerouted in the above process. Suppose that \( P \) ends at the vertex \( c' \) in \( S \) that is only adjacent to the center \( c \) in \( \sigma(j) \). By possibly relabeling the index \( j \), we may assume that each node \( \sigma(j) \) does not hit any path in \( \mathcal{P} \) and any path in \( \mathcal{P}' \) for \( j \geq 8k^2 + 1 \), and moreover \( j_1 \leq 6k^2 \). Thus, we may assume that none of the nodes \( \sigma(j) \) with \( j \geq 8k^2 + 1 \) has a neighbor to \( c \).

By the definition of the center, since \( p \geq 30k^2 \), it follows that there are two indices \( j_3, j_4 \geq 8k^2 + 1 \) such that the node \( \sigma(j_3) \) has a neighbor in one component \( T_1 \) of \( \sigma(j_1) - c \) that hits some path in \( \mathcal{P} \) that is rerouted at the node \( \sigma(j_1) \), and the node \( \sigma(j_4) \) has a neighbor in the other component \( T_2 \) of \( \sigma(j_1) - c \) that hits another path in \( \mathcal{P} \) that is also rerouted at the node \( \sigma(j_1) \). Hence at least two paths in \( \mathcal{P} \) are already rerouted at the node \( \sigma(j_1) \).

It follows that the number of the rerouted paths in \( \mathcal{P} \) is at least twice as many as the number of paths in \( \mathcal{P} \) that are not rerouted, and hence \(| \mathcal{P}' | \geq 2k^2 \).

Moreover, since \(| \mathcal{P} | = 3k^2 \), our above rerouting procedure implies that we may assume that none of the nodes \( \sigma(j) \) hits a path in \( \mathcal{P}' \) for \( j \geq 10k^2 \). Therefore, we may assume that there are 7\( k^2 \) odd \( S \)-paths \( P'_{6k+1} \in \mathcal{P}' \) with endpoints \( c'_{12k^2} \cap c'_{12k^2+2} \) that do not intersect any vertex in \( \mathcal{P}' \) for \( i = 1, \ldots, 7k^2 \). Let us observe that \( \sigma(2i-1) \cup \sigma(2i) \cup P_i \) for \( i = 6k^2 + 1, \ldots, 13k^2 \) (where \( P_i \) is an odd path in \( \mathcal{P}' \)) gives rise to an odd \( K_{7k^2} \)-model, see Theorem 4.1.

Since \( 2k \) vertex-disjoint \( 1.5k \)-spiders are given, and since after deleting the vertices in \( S \) from \( G' \), any \( k \)-spider in \( G' \) contains a \((k-1)\)-spider with endpoints in \( C \) in \( G \), it follows that:

\[ \text{[\textsuperscript{[**]}] there are } k \text{ vertex-disjoint } (k-1)\text{-spiders} \]

\[ L \text{ in } G \text{ (not in } G') \text{, such that, for } i = 10k^2 + k + 1, \ldots, 11k^2 \text{ the node } \sigma(i) \text{ contains a vertex } u_i \text{ of degree one in the spiders, but does not intersect any other vertex in the } k \text{-vertex-disjoint } (k-1)\text{-spiders} \ L \text{.} \]

Let us consider the graph \( R \) obtained from \( \sigma(2i-1) \cup \sigma(2i) \cup P_i \) for \( i = 6k^2 + 1, \ldots, 13k^2 \) and \( \sigma(j) \) for \( j = 10k^2 + k + 1, \ldots, 11k^2 \). Thus \( R \) contains an odd \( K_{7k^2} \)-model \( L' \). Let \( U = \{u_{10k^2+k+1}, \ldots, u_{11k^2}\} \).

We claim that the graph \( R \) and the terminals \( U \) satisfy the assumption of Theorem 3.4 (with \( G = R \) and \( U = S \) in Theorem 3.4). For, suppose there is a separation \((A, B)\) of order at most \( k(k-1) - 1 \) in \( R \) such that a node \( \sigma(i') \) in \( R \) is contained in \( B - A \) and all the vertices of \( U \) are in \( A \). Since we have the odd \( K_{7k^2} \)-model \( L' \), thus none of the nodes of \( L' \) is contained in \( A - B \). However, the existence of the \( k \)-vertex-disjoint \((k-1)\)-spiders satisfying [\textsuperscript{[**]}] implies that at least one of the nodes of \( L' \) is contained in \( A - B \), a contradiction. Thus the graph \( R \) and the terminals \( U \) satisfy the assumption of Theorem 3.4. Therefore, by Theorem 3.4, we can find \( k(k-1)/2 \) disjoint paths with specified endpoints in \( U \) and with any prescribed parity (i.e., even or odd) in \( R \). By choosing the appropriate labeling of the endpoints of the parity disjoint paths in \( R \), together with the \( k \) vertex-disjoint \((k-1)\)-spiders \( L \), we can find a totally odd \( K_k \)-subdivision in \( G \).

Let us observe that the proofs of Lemmas 5.2 and 5.3 can be easily translated into a polynomial time algorithm, if the even \( K_r \)-model, the \( 2k \) vertex-disjoint \( 1.5k \)-spiders with \( A = V(G') - S \) and \( B = S \), and \( 7k^2 \) disjoint odd \( S \)-paths as in Hypothesis are given. In fact, the odd \( S \) paths can be found in polynomial time by Theorem 2.1, if the paths exist. Moreover, the \( 2k \) vertex-disjoint \( 1.5k \)-spiders with \( A = V(G') - S \) and
$B = S$ can be found in polynomial time by Theorem 5.1 if they exist.

6 Structure Theorems

In the next section, we shall give a structure theorem for graphs without a totally odd $K_3$-subdivision. In order to do that, we need to give several definitions concerning Robertson-Seymour's graph minor structure theorem.

6.1 Tangles Let $G$ be a graph. A tangle of order $k$ of $G$ is a set $\mathcal{T}$ of separations of $G$ of order $< k$ satisfying the following three conditions.

1. For all separations $(A, B)$ of order $< k$, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$.
2. If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$.
3. If $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Note that if $(A, B) \in \mathcal{T}$ then $(B, A) \notin \mathcal{T}$; we think of $B$ as the 'big side' of the separation $(A, B)$, with respect to this tangle (and similarly $A$ is the "small side").

Let $\mathcal{T}$ be a tangle of order at least $p$. We say that a $K_p$-model is controlled by the tangle $\mathcal{T}$ if no node of the $K_p$-model is contained in $A - B$ of any separation $(A, B) \in \mathcal{T}$ of order at most $p - 1$.

6.2 Societies and Vortices A society is a pair $(G, \Omega)$, where $G$ is a graph and $\Omega$ a cyclic permutation of a subset $V(\Omega)$ of $V(G)$ (we call $V(\Omega)$ society vertices). Note that for every $w \in V(\Omega)$ we have $V(\Omega) = \{\Omega(w) \mid 0 \leq j < |V(\Omega)|\}$. The length of a society $(G, \Omega)$ is $|V(\Omega)|$.

A society $(G, \Omega)$ of length $\ell$ is a $p$-vortex if for all $w \in V(\Omega)$ and $k \in [\ell]$ there do not exist $(p + 1)$ mutually disjoint paths of $G$ between $\{\Omega(w) \mid 0 \leq j < k\}$ and $\{\Omega(w) \mid k \leq j < \ell\}$.

A linear decomposition of a society $(G, \Omega)$ of length $\ell$ is a sequence $(X_i)_{0 \leq i < \ell}$ of subsets of $V(G)$ such that

1. $\bigcup_{i=0}^{\ell-1} X_i = V(G)$.
2. $X_i \cap X_k \subseteq X_j$ for $0 \leq i, j, k < \ell$.
3. There is a $x_0 \in V(G)$ such that $\Omega(x_0) \in X_i$ for $0 \leq i < \ell$.

The width of the linear decomposition $(X_i)_{0 \leq i < \ell}$ is $\max\{|X_i| \mid 0 \leq i < \ell\}$, and the depth of $(X_i)_{0 \leq i < \ell}$ is $\max\{|X_i \cap X_{i+1}| \mid 0 \leq i < \ell - 1\}$. Sometimes $X_i$ is called a bag (of a linear decomposition of a society $(G, \Omega)$).

The following is proved in [35].

Theorem 6.1. If a society $(G, \Omega)$ is a $p$-vortex then it has a linear decomposition of depth at most $p$.

6.3 Near Embeddings Robertson and Seymour's main theorem is concerning the structure capturing a big side with respect to a tangle. We now mention one version of their result.

For a positive integer $\alpha$, a graph $G$ is $\alpha$-nearly embeddable in a surface $\Sigma$ if there is a subset $Z \subseteq V(G)$ with $|Z| \leq \alpha$, two sets $V = \{(G_1, \Omega_1), \ldots, (G_n, \Omega_n)\}$, where $\alpha < \alpha'$, and $W = \{(G_{\alpha'}, \Omega_{\alpha'+1}), \ldots, (G_n, \Omega_n)\}$ of societies, and a graph $G_0$ such that the following conditions are satisfied.

1. $G = G_0 \cup G_1 \cup \ldots \cup G_n$.
2. For all $i \in [n]$ we have $E(G_0) \cap E(G_i) = \emptyset$ and $V(G_0) \cap V(G_i) \subseteq V(\Omega_i)$. For all distinct $i, j \in [n]$ we have $E(G_i \cap E(G_j) = \emptyset$ and $V(G_i \cap V(G_j) \subseteq V(\Omega_i) \cap V(\Omega_j)$. Furthermore, if $i, j \leq \alpha'$ then $V(G_i) \cap V(G_j) = \emptyset$.
3. Each $(G_i, \Omega) \in V$ is an $\alpha$-vortex (for simplicity, $(G_i, \Omega_i)$ is called a vortex). By Theorem 6.1, $(G_i, \Omega_i)$ has a linear decomposition of depth at most $\alpha$.
4. Each $(G_i, \Omega_i) \in W$ has length at most $3$.
5. There are closed disks $\Delta_1, \ldots, \Delta_n \subseteq \Sigma$ with disjoint interiors and an embedding $\sigma : G_0 \hookrightarrow \Sigma$ such that for all $i \in [n]$ we have $\sigma(G_0) \cap \operatorname{int}(\Delta_i) = \emptyset$ and $\sigma(V(G_0)) \cap \operatorname{bd}(\Delta_i) = \sigma(V(\Omega_i))$, and the cyclic ordering of the vertices in $\sigma(V(\Omega_i))$ induced by $\Omega_i$ is compatible with the natural cyclic ordering of the vertices on the simple closed curve $\operatorname{bd}(\Delta_i)$.

We call $(\sigma, G_0, Z, V, W)$ an $\alpha$-near embedding of $G$ in $\Sigma$ or just near-embedding if the bound is clear from the context.

Let $G_0'$ be the graph resulting from $G_0$ by joining any two nonadjacent vertices $u, v \in G_0$ that lie in a common vortex $V \in W$; the new edge $uv$ of $G_0'$ will be called a virtual edge. By embedding these virtual edges disjointly in the disks $\Delta$ accommodating their vortex $V$, we extend our embedding $\sigma : G_0 \hookrightarrow \Sigma$ to an embedding $\sigma' : G_0' \hookrightarrow \Sigma$. We shall not normally distinguish $G_0'$ from its image in $\Sigma$ under $\sigma'$.

A near-embedding $(\sigma, G_0, Z, V, W)$ is $\mathcal{T}$-central, for a tangle $\mathcal{T}$ of $G$, if for all $(H, \Omega) \in V \cup W$ there is no $(A', B') \in \mathcal{T}$ with $B' \subseteq H$.

We are now ready to mention the "main" result of the graph minor theory in [37] (see (3.1) in [37]).

Theorem 6.2. For every graph $R$ there are integers $\alpha = \alpha(R)$ and $\omega = \omega(R)$ such that the following holds. Every graph $G$ with a tangle $\mathcal{T}$ of order at least $\omega$ either controls $R$ as a minor or has a $\mathcal{T}$-central $\alpha$-near embedding in some surface $\Sigma$ in which $R$ cannot be embedded.
Given a tangle $\Sigma$, a polynomial time algorithm to construct one of the conclusions in Theorem 6.2 is given in [9, 18, 27].

By using Theorem 6.2, the following result which strengthens Theorem (1.3) of [37] can be shown (see Section 3 and (1.3) in [37]). See the proof in [12].

**Theorem 6.3.** For every graph $R$ there exist integers $\alpha$ and $\theta$ such that for every graph $G$ that does not contain $R$ as a minor and every $Z \subseteq V(G)$ with $|Z| \leq 3\theta - 2$ there is a tree-decomposition $(V_t)_{t \in T}$ of $G$ with a rooted tree $T$ and root $t''$, such that for every $t \in T$, there is a surface $\Sigma_t$ in which $R$ cannot be embedded, and the torso of $G_t$ (i.e., obtained from the graph induced by $V_t$ by making all $G_t \cap G_{t'}$ cliques for $t' \in T$, where $t'$ is a children of $t$) has an $\alpha$-near embedding $(\sigma_t, G_{t,0}, Z_t, V_t, \emptyset)$ into $\Sigma_t$ with the following properties:

1. All vortices have linear decompositions of width at most $\alpha$.
2. For every $t' \in T$ with $t' \in E(T)$ there is a vertex set $X$ which is either
   (a) two consecutive parts of an $\alpha$-vortex or
   (b) a subset of $V(G_{t,0})$ that induces in $G_{t,0}$ a $K_1$, a $K_2$ or a boundary triangle (i.e., it bounds a disk that is a face of $G_{t,0}$ in $\Sigma_t$).
   such that $V_t \cap V_{t'} \subseteq X \cup Z_t'$. Further, $Z \subseteq Z_t'$.

A polynomial time algorithm to construct such a tree-decomposition in Theorem 6.3 is given in [9, 18, 27].

7 Structure theorem for graphs without a totally odd $K_t$-subdivision

We now give a structure theorem for graphs without a totally odd $K_t$-subdivision. For technical reason, we shall prove the following structure theorem.

**Theorem 7.1.** For every integer $k$ there exist integers $\alpha$ and $\theta$ such that for every graph $G$ that does not contain a totally odd $K_t$-subdivision and every $Z \subseteq V(G)$ with $|Z| \leq 3\theta - 2$ there is a rooted tree-decomposition (with a rooted tree $T$ and root $t''$) $(V_t)_{t \in T}$ of $G$ such that for every $t \in T$, either

1. there is a vertex set $Z_t'$ (an apex set) of order at most $2$ such that $V_t - Z_t'$ induces a bipartite graph, and moreover, $| V(t) \cap V(t') - Z_t'| \leq 1$ for each $t' \in T$ where $t'$ is a children of $t$, or
2. there is a vertex set $Z_t'$ (an apex set) of order at most $\alpha$ such that $V_t - Z_t'$ induces a 6$k$-degenerate graph (i.e, any induced subgraph has a vertex of degree at most $6k$), and moreover, $| V(t) \cap V(t') - Z_t'| \leq 1.5k - 1$ for each $t' \in T$ where $t'$ is a children of $t$, or

3. there is a surface $\Sigma_t$ of Euler genus $\alpha$, and the torso of $G_t$ (i.e., obtained from the graph induced by $V_t$ by making all $G_t \cap G_{t'}$ cliques for $t' \in T$, where $t'$ is a children of $t$) has an $\alpha$-near embedding $(\sigma_t, G_{t,0}, Z_t, V_t, \emptyset)$ into $\Sigma_t$ with the following properties:
   (a) All vortices have linear decompositions of width at most $\alpha$.
   (b) For every $t' \in T$ with $t' \in E(T)$ there is a vertex set $X$ which is either
      i. two consecutive parts of an $\alpha$-vortex or
      ii. a subset of $V(G_{t,0})$ that induces in $G_{t,0}$ a $K_1$, a $K_2$ or a boundary triangle (i.e., it bounds a disk that is a face of $G_{t,0}$ in $\Sigma_t$).
   such that $V_t \cap V_{t'} \subseteq X \cup Z_t'$.

Further, $Z \subseteq Z_{t'}$, Moreover, if 2 happens, then $G$ contains an odd $K_{6k^2+4k^3}$-model.

In addition, given $k$, we can find either a totally odd $K_k$-subdivision or such a tree-decomposition (and an odd $K_{6k^2+4k^3}$-model, if 2 happens) in polynomial time.

We note that the same conclusion of Theorem 7.1 is also true if we replace “totally odd” by “parity”. Hence this generalizes the structure theorem for subdivision-free graphs [17, 29].

**Proof:** Let $r \geq 23k^4$ and $\Theta \geq 24k^3$ Applying Theorem 6.2 with a given graph $K_r$ yields two constants $\alpha$ and $\theta$. Let $\theta := \max(\theta, 3k + 1)$ and $\alpha := 4\theta - 2$.

The proof proceeds by induction on $|G|$. We may assume that $|Z| = 3\theta - 2$, since if it is smaller we add arbitrary vertices to $Z$. Note, we may assume that such vertices exist, as the theorem is trivial for $|G| < \alpha$.

We may assume that

(7.1) There is no separation $(A, B)$ of order at most $\theta$ such that both $|Z \setminus A|$ and $|Z \setminus B|$ are of size at least $|A \cap B|$.

Otherwise, let $Z_A := (A \cup Z) \cup (A \cap B)$. By assumption, $|A \cap B| \leq |Z \setminus A|$ and therefore, $|Z_A| \leq |Z|$. We apply our theorem inductively to $A$ and $Z_A$, which yields a tree-decomposition of $A$ with one part $T_A$ such that the apex set of $T_A$ contains $Z_A$. Similarly, we apply the theorem to $B$ and $Z_B := (B \cap Z) \cup (A \cap B)$. We combine these two tree-decompositions by joining a new part $Z \cup (A \cap B)$ to both $T_A$ and $T_B$ and obtain a tree-decomposition of
We now consider Case 1. By Lemma 5.3, there exists a graph $G'$ such that $G = G' \setminus Z$, where $Z$ is a set of order $\theta$ vertices. We deduce further, that for every $(A, B) \in T$, the small side $A$ contains less than $\theta$ vertices from $Z$. Hence, the union of three small sides cannot be $V(G)$ as it contains at most $3\theta - 3$ vertices from $Z$, which shows property (ii) and proves (7.2).

From (7.1) and the definition of $T$ we conclude that $|(A \setminus B) \cap Z| < |A \cap B|$ for every $(A, B) \in T$.

We now apply Theorem 6.2 with this tangle $T$, which gives us either

**Case 1** a $K_p$-model controlled by the tangle $T$, or

**Case 2** a $T$-central $\delta$-near embedding $(\sigma, G_0, A, \hat{V}, \hat{W})$ of $G$ in some surface $\Sigma$.

Moreover, as remarked just after Theorem 6.2, we can obtain one of Cases 1 and 2 in polynomial time.

Let us consider each of Cases 1 and 2.

**Case 1.** As in the proof of Theorem 4.2, there is an even $K_p$-minor $L$ (with $p \geq 2^{4k^2}$). Moreover, such an even clique minor can be found in polynomial time (in fact, in $O(n)$ time). Since each node of $L$ consists of node(s) of the $K_p$-minor, thus $L$ is also controlled by the tangle $T$.

Let $N_1, \ldots, N_p$ be the nodes of this even $K_p$-model $L$, and let $C_1, \ldots, C_p$ be centers of $N_1, \ldots, N_p$, respectively. By Theorem 4.2, either

1. for any $i, j$ with $i \neq j$, $|W_i \cap W_j| \leq 1.5k - 1$, and
2. subject to that, $q$ is as big as possible.

Since $G - X$ has no 1.5k-spider with respect to $C$, for each $i \leq q$, there is a separation $(A_i, B_i)$ of order at most $1.5k - 1$ in $G - X$ such that $W_i$ is in $A_i$ and $C - X$ is $B_i$. We observe the following.

There is no node of the even clique model $L$ that is contained in $A_i - B_i$.

We take such separations $(A_i, B_i)$ such that $|A_i \cap B_i|$ is as small as possible, and subject to that,

$$\sum_{i=1}^{q} |A_i|$$

is minimum.

We assume that for any two $i, j$ with $i \neq j$, $A_i - A_j \neq \emptyset$ and $A_i - A_j \neq \emptyset$. Hence $(B_i - A_i) \cap (A_j \cap B_j) \neq \emptyset$ and $(B_j - A_j) \cap (A_i \cap B_i) \neq \emptyset$ (otherwise, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$, which is irrelevant for our purpose).

We claim that for any $i, j$ with $i \neq j$,

$$W_i \cap (A_j - B_j) = \emptyset$$

and

$$W_j \cap (A_i - B_i) = \emptyset.$$

To prove the claim suppose to the contrary that $W_i \cap (A_j - B_j) \neq \emptyset$. Since $|A_i \cap B_i| \leq 1.5k - 1$ and $W_i$ is 1.5k-connected, hence $W_i$ is in $A_1 \cap A_2$.

Let

$$A_1' = A_1 \cap A_2,$$

$$B_1' = B_1 \cup B_2,$$

$$A_2' = A_1 \cup A_2,$$

$$B_2' = B_1 \cap B_2.$$
Then \((A'_1, B'_1)\) is a separation of \(G - X\) with \(W_1 \subseteq A'_1\). Moreover, \((A'_2, B'_2)\) is a separation of \(G - X\) with \(W_1 \subseteq A'_2\). We have

\[|A_1 \cap B_1| + |A_2 \cap B_2| = |A'_1 \cap B'_1| + |A'_2 \cap B'_2|\]

By our choice, \(|A'_1 \cap B'_1| > |A_1 \cap B_1|\) and \(|A'_2 \cap B'_2| > |A_1 \cap B_1|, |A_2 \cap B_2|\). This is a contradiction. Thus the claim holds.

Hence we can choose \(W_i\) and a separation \((A_i, B_i)\) for \(i = 1, \ldots, q\) such that

for any \(i, j\) with \(i \neq j\), \(W_i \cap (A_i \cap A_j) = \emptyset\) and \(W_j \cap (A_i \cap A_j) = \emptyset\).

We now claim that \((A_1 \cap A_j) - (B_1 \cap B_j) = \emptyset\) for any \(i, j\) with \(i \neq j\). Suppose \((A_1 \cap A_2) - (B_1 \cap B_2) \neq \emptyset\). Let

\[A''_1 = A_1 \cap B_2,\]
\[B''_1 = A_2 \cup B_1,\]
\[A''_2 = A_2 \cap B_1,\]
\[B''_2 = A_1 \cup B_2.\]

Then \((A''_1, B''_1)\) is a separation of \(G - X\) with \(W_1 \subseteq A''_1\), and \((A''_2, B''_2)\) is a separation of \(G - X\) with \(W_2 \subseteq A''_2\). By our choice, \((A_1 \cap A_2) - (B_1 \cap B_2) \neq \emptyset\), \(|A''_1 \cap B''_1| > |A_1 \cap B_1|\) and \(|A''_2 \cap B''_2| > |A_2 \cap B_2|\). This is a contradiction, since

\[|A_1 \cap B_1| + |A_2 \cap B_2| = |A''_1 \cap B''_1| + |A''_2 \cap B''_2|.\]

Therefore \(A_i \cap A_j = \emptyset\) for any \(i, j\) with \(i \neq j\).

Since the even clique model \(L\) is controlled by the tangle \(T\), thus \((A_i \cup X, B_i \cup X) \in T\) for \(1 \leq i \leq q\). Therefore, \(|X| + |A_i \cap B_i| + |Z - X - B_i| \leq 3\Theta - 2\), and hence we can apply induction to each \(A_i \cup X\) with \(Z = X \cup (A_i \cap B_i) \cup (Z - X - B_i)\) for \(1 \leq i \leq q\).

Let \(W = \bigcap_{i=1}^{q} B_i\). Then by the maximality of \(q\), \(W\) does not contain a 1.5k-connected subgraph, and hence it is 6k-degenerate by the result of Mader [32]. Since \(W \cup X \cup Z\) satisfies one of the structures in Theorem 7.1 (by putting \(\gamma' = Z \cup X\). Note that \(|Z \cup X| \leq \alpha\) since \(|X| \leq 2^{3k}\) and \(\Theta \geq 2^{3k^3}\), thus we obtain a tree-decomposition as in the conclusion of Theorem 7.1 by putting the tree-decompositions of \(A_1 \cup X, \ldots, A_q \cup X\), at \(W \cup X \cup Z\). Note that \(W \cup X \cup Z\) is a root piece of the resulting tree-decomposition.

Let us observe that we can find the vertex set \(X\) in polynomial time by Theorem 5.1. In order to find a separation \((A_i, B_i)\) for \(i = 1, \ldots, q\), firstly we shall find a 1.5k-connected subgraph \(W_i\). Then we find such a separation \((A_i, B_i)\) in \(G - X\). Finding such a graph \(W_i\) can be done in polynomial time by following the proof of Mader [32], and finding such a separation \((A_1, B_1)\) can be also done in polynomial time (in fact, \(O(m)\) time) by the standard max-flow min-cut algorithm. Then we find a 1.5k-connected subgraph \(W_k\) in \(B_1\), and find such a separation \((A_2, B_2)\) in \(G - X\), and so on. This can be done in polynomial time, since the graph \(B_i\) is smaller than \(B_{i-1}\) by our construction.

Since we can find a desired tree-decomposition for each smaller graph in polynomial time, hence in Case 1.1, we can construct a tree-decomposition as in the conclusion of Theorem 7.1 in polynomial time.

In addition, we can find an odd \(K_{2k^2 + 4k}\)-model by the assumption of Case 1.1.

We now look at Case 1.2. By Theorem 4.2, \(G\) has a vertex set \(X\) of order 4q such that \(G - X\) can be written as \(G_0 \cup G_1 \cup \cdots \cup G_t\) (for some \(t\)) with the following property:

1. \(G_0\) has a \(K = K_{p - 4q}\)-model (indeed at least \(p - 4q\) nodes of \(L\)), and is bipartite,
2. \(|G_i \cap G_j| \leq 1\) for any \(1 \leq i < j \leq t - 1\), and \(|G_i \cap G_i| = 0\) for any \(i\),
3. for \(i = 1, \ldots, t - 1\), \(G_i\) is a connected subgraph of \(G - X\) and \(|G_i \cap G_0| = 1\), and
4. \(G_t\) is a subgraph of \(G - X\) (not necessarily connected) such that each vertex in \(V(G_t)\) has no neighbors in \(G_i\) for \(i = 0, \ldots, t - 1\).

Since the even clique model \(L\) is controlled by the tangle \(T\),

for \(i = 1, \ldots, t\), \(|Z \cap (V(G_0) \cup X)| \geq |(V(G_0) \cup X) \cap Z|\), and hence \(|(V(G_i) \cup X) \cap Z|\) contains at most \(3/2\Theta + 4q\) vertices of \(Z\).

Thus we can apply induction to \(G_i \cup X\) with \(Z_i = ((V(G_i) \cup X) \cap Z) \cup X \cup (V(G_i) \cap V(G_0))\) for \(i = 1, \ldots, t\). Note that \(|Z_i| \leq 3\Theta - 2\), so the induction hypothesis is satisfied. Since \(G_0 \cup U \cup Z\) satisfies one of the structures in Theorem 7.1 (Note that \(|Z \cup X| \leq \alpha\) since \(|X| \leq 2^{3k^3}\) and \(\Theta \geq 2^{3k^3}\) ), thus we obtain a tree-decomposition as in the conclusion of Theorem 7.1, by putting the tree-decompositions of \(G_1 \cup X, \ldots, G_t \cup X\) at \(G_0 \cup U \cup Z\). Note that \(G_0 \cup U \cup Z\) is a root piece of the resulting tree-decomposition.

Let us observe that since we can find a desired tree-decomposition for each smaller graph in polynomial time (and in addition, we can find an odd \(K_{2k^2 + 4k}\)-model in a smaller graph), the above structure can be found in polynomial time by Theorem 4.2. Thus in Case 1.2, we can construct a tree-decomposition as in the conclusion of Theorem 7.1 in polynomial time.
Case 2. At a high level, our plan is now to split up $G$ at separators consisting of apex vertices, society vertices $\Omega(V)$ for $V \in \mathcal{W}$ and vertices of single parts of linear decompositions of vertices in $\hat{V}$. We obtain a part that contains $G_0$ and which we know how to embed $\alpha$-nearly; this part is going to be one part of a new tree-decomposition (in fact, it would be the root). We find tree-decompositions for all subgraphs of $G$ that we split off inductively and eventually combine these tree-decompositions to a new one that satisfies our theorem.

Let us consider $(G_i, \Omega_i) \in \mathcal{W}$. Our embedding is $T$-central, therefore the separation $(V(G_i) \cup \hat{A}, V(G \setminus G_i) \cup \hat{A})$, whose order is smaller than $3 + |A| \leq \theta$, lies in $T$. By (7.3), $G_i$ contains less than $\theta$ vertices of $Z$. Thus, $Z' := \Omega \cup \hat{A} \cup (Z \cap G_i)$ contains at most $3 + \alpha + \theta \leq 3\theta-1$ vertices. We apply our theorem inductively to the smaller graph induced by $V(G_i) \cup \hat{A}$ with $Z'$. Let $H^i$ be a part of the resulting tree-decomposition $(T^i, H^i)$ that accommodates $Z'$.

For every vertex $(G_i, \Omega_i) \in \hat{V}$ with $\Omega_i = \{w_1, \ldots, w_{n(i)}\}$ let us choose a linear decomposition $(X^i_1, \ldots, X^i_{n(i)})$ of depth at most $\alpha$. We define

$$X^i_j := \begin{cases} (\hat{X}^i_j \cup \hat{X}^i_j) \cup \{w_i\} & \text{for } j = 1 \\ (\hat{X}^i_j \cup (\hat{X}^i_{j-1} \cup \hat{X}^i_{j+1})) \cup \{w_j\} & \text{for } 1 < j < n(i) \\ (X^i_{n(i)} \cup X^i_{n(i)-1}) \cup \{w_{n(i)}\} & \text{for } j = n(i) \end{cases}$$

By $G^i_j$ we denote the graph on $X^i_1 \cup \ldots \cup X^i_{n(i)}$ where every $X^i_j$ induces a complete graph but no further edges are present. Now, as the depth of $(G_i, \Omega_i)$ is at most $\alpha$, every $X^i_j$ contains at most $2\alpha + 1$ vertices and thus, $(X^i_1, \ldots, X^i_{n(i)})$ is a linear decomposition of the vortex $V^-_i := (G^-_i, \Omega_i)$ of width at most $2\alpha + 1$. Let $V$ denote the set of these new vertices.

For every $j = 1, \ldots, n(i)$, the pair

$$(\hat{X}^i_j \cup \hat{A}, (V(G) \setminus (\hat{X}^i_j \cup \hat{X}^i_{j+1})) \cup \hat{A})$$

is a separation of order at most $|X^i_j \cup \hat{A}| \leq 2\alpha + 1 + \alpha \leq \theta$. As before, our embedding is $T$-central and thus, the separation lies in $T$. By (7.3), at most $\theta - 1$ vertices from $Z$ lie in $\hat{X}^i_j$. Let $Z' := X^i_j \cup \hat{A} \cup (\hat{X}^i_j \setminus \hat{X}^i_j)$. This set contains at most $3\theta - 1$ vertices and, similar to before, we can apply our theorem inductively to the smaller graph induced by $\hat{X}^i_j \cup \hat{A}$ with $Z'$. We obtain a tree-decomposition $(T^i_j, H^i_j)$ of this graph, with one part $H^i_j$ accommodating $Z'$.

Now, with $V_0 := G_0 \cup \hat{A}$, we can write

$$G = V_0 \cup (\bigcup \mathcal{W}) \cup (\bigcup \{X^i_j : V_i \in \mathcal{V}, 1 \leq j \leq n(i)\}).$$

By induction, we obtained tree-decompositions for all vertices in $\mathcal{W}$ and all the graphs $X^i_j$ with the required properties. We can now construct a tree-decomposition of $G$: We just add a new vertex $v_0$ representing $V_0$ to the union of all the trees $T^i$ and $T^i_j$ and add edges from $v_0$ to every vertex representing an $H^i$ or an $H^i_j$ we found in our proof.

We still have to check that the torso of the new part $V_0$ can be $\alpha$-nearly embedded as desired. But this is easy: Let $G'_0$ be the graph resulting from $G_0$ if we add an edge $xy$ for every two nonadjacent vertices $x$ and $y$ that lie in a common vertex $V \in \mathcal{W}$. We can extend the embedding $\sigma : G_0 \hookrightarrow \Sigma$ to an embedding $\sigma' : G'_0 \hookrightarrow \Sigma$ by mapping the new edges to the discs $D(V)$. Then, $G' := G'_0 \cup G^-_1$ is the torso of $V_0$ in our new tree-decomposition and with $(\sigma', G'_0, \hat{A} \cup Z, \mathcal{V}, \emptyset)$ we have an $\alpha$-near embedding of $G'$ in $\Sigma$ whose apex set contains $Z$.

Let us observe that Case 2 can be also done in polynomial time, as we just need to get tree-decompositions of all the following smaller graphs together:

$$V_0 \cup (\bigcup \mathcal{W}) \cup (\bigcup \{\hat{X}^i_j : V_i \in \mathcal{V}, 1 \leq j \leq n(i)\}).$$

Since we can find a desired tree-decomposition for each smaller graph in polynomial time (and in addition, we can find an odd $K_{3k+2}$-model in a smaller graph), we can obtain a tree-decomposition for $G$ with desired properties in Theorem 7.1.

Remark 1. By the above proof, we can also add the following conclusion to Theorem 7.1

If 1 of Theorem 7.1 happens, then there is a $K_{2k+4}$-model in $G$ (and moreover, we can find such a model in polynomial time).

Remark 2. It is easy to see that our whole proof would also work if we replace a “totally odd $K_k$-subdivision” by a “parity $K_k$-subdivision” in Theorem 7.1.

Remark 3. In [17], Grohe and Márxi showed that every graph with no $K_k$-subdivision has a tree-decomposition such that each piece is either

1. after deleting bounded number of vertices, an “almost” embedded graph into a bounded-genus surface, or
2. after deleting bounded number of vertices, a graph with maximum degree at most $f(k)$ for some function $f$ of $k$.

We can replace 2 of Theorem 7.1 by the second conclusion of Grohe and Márxi’s theorem by following their proof in [17]. Since we only need to replace Case 1.1 by their arguments, so we omit the proof.


**Appendix**

8 Proof of Theorem 1.3

In this section, we shall give a sketch of the proof for Theorem 1.3. We prove Theorem 1.3 by induction on the number of vertices. Our polynomial time algorithm would follow from our constructive proof. We may assume that $\chi(G) \geq 3$ (otherwise, $G$ is bipartite, and we can easily color all the vertices of $G$ using two colors).

Let $G$ be a graph such that Theorem 1.3 holds for every graph with order at most $|V(G)| − 1$. Hence $G$ is a graph with no totally odd $K_4$-subdivision. Let $S$ be a vertex set of $G$ with at most $6(k−1)$ vertices. We are also given a precoloring of $S$.

First, we derive some easy properties.

1. Each vertex in $G − S$ has degree at least $6k$.

Since $2\chi(G) + 6(k−1) ≥ 6k$, it is easy to see that any coloring of $G − v$ can be extended to a coloring of $G$, if degree of $v$ is at most $6k − 1$.

2. There is no separation $(A, B)$ of order at most $3(k−1)$ such that both $A − B$ and $B − A − S$ are nonempty.

Suppose such a separation $(A, B)$ exists. Since $|S| ≤ 6(k−1)$, thus one of $|A \cap B| + |(A − B) \cap S|$ and $|A \cap B| + |(B − A) \cap S|$ is at most $6(k−1)$. Without loss of generality, $|A \cap B| + |(B − A) \cap S| ≤ 6(k−1)$.

Since $B − A − S$ is nonempty, $|A \cup S|$ is smaller than $|G|$. We can therefore apply induction first to $A \cup S$ with $S$ precolored. Then we obtain a coloring of $A \cap B$, which has at most $3(k−1)$ vertices by our assumption. Let $S' = (A \cap B) \cup (B − A) \cap S$. We apply induction to $B$ with $S'$ precolored. The precoloring of $S'$ comes from the coloring of $A \cup S$. Thus we can combine the colorings of $B$ and $A$ to obtain a coloring of the whole graph $G$ that extends the precoloring of $S$. This proves (2).

Let us apply Theorem 7.1. We first consider the case when there is an odd $K_{2k^2+4k}$-model $L$. Let $p = 2k^2+4k$. The centers $C = \{c_1, \ldots, c_p\}$ can be defined for $L$ as before. By Lemma 5.3, we may assume that there are no $2k$ vertex-disjoint $1.5k$-spiders with respect to $C$ (for otherwise, we can find a totally odd $K_{4}$-subdivision). Hence by Theorem 5.1, there is a vertex set $X$ of order at most $2k^2$ such that $G − X$ has no $1.5k$-spider with respect to $C$.

Let us take $X'' \subseteq X$ with $|X''| = k$. For each vertex $v \in X''$, we add $k − 2$ copies $v_1, \ldots, v_k−2$ to $G$ such that $N_G(v_i) = N_G(v)$ for $i = 1, \ldots, k − 2$. Let $G'$ be the resulting graph, and $X'$ be the resulting vertex set of $X''$. Hence $|X'| ≤ (k−1)k$. If $S = X'$ and the odd clique model $L$ satisfy the hypothesis of Theorem 3.4 in $G'$, then we can easily find a totally odd $K_{4}$-subdivision.

This implies that there is a separation $(A', B')$ of order at most $(k−1)l$ (with $l ≤ k$) such that $G'' = B''$ contains $l$ vertices in $X$ and $B'' = A''$ contains a node of $L$. By continuing this procedure, we obtain a separation $(A, B)$ of order at most $2k^2+k$ such that all but at most $k − 1$ vertices in $X$ are contained in $A − B$, and at least one node of $L$ is contained in $B − A$. Since $|A \cap B| ≤ 2k^2+k$, this implies that all the nodes of $L$ must hit $B$. Let $X \subseteq X$ be a set of at most $k − 1$ vertices in $X$ that are contained in $A \cap B$. We now follow the proof of Case 1.1 in Theorem 7.1. We claim that there are at most $2k^2+k$ indices $i$ such that $(A_i, B_i)$ is a separation of order at most $1.5k − 1$ in $G − X$ as in the proof of Case 1.1 in Theorem 7.1 and moreover, $A_i − B_i$ contains a vertex in $B − A$ (as in the proof of Case 1.1 in Theorem 7.1, $(A_i, B_i) \cap \{A_i − B_i\} = \emptyset$). For otherwise, since $|A \cap B| ≤ 2k^2+k$ and hence there are at most $2k^2+k$ indices $i$ such that $A_i − B_i$ contains a vertex in $A \cap B$, there is an index $i$ such that $(A_i \cup X, B_i \cup X)$ is a separation of order at most $2.5k − 1$ in $G$ such a way that $A_i$ is contained in $B_i$, and $A_i − B_i − S \neq \emptyset$ but this contradicts (2).

We take all the indices $i$ such that $(A_i, B_i)$ is a separation of order at most $1.5k − 1$ in $G − X$ as in the proof of Case 1.1 in Theorem 7.1 and moreover, $A_i − B_i$ contains a vertex in $B − A$. By the previous remark, as in the proof of Case 1.1 in Theorem 7.1, $(A_i, B_i) \cap \{A_i − B_i\} = \emptyset$ and there are at most $2k^2+k$ indices $i$. Let $R = \bigcup_i (A_i − B_i).$ By the remark in the proof of Case 1.1 in Theorem 7.1, none of the nodes of $L$ is contained in $A_i − B_i$, and hence in $R$. Let $R' = \bigcup_i (A_i \cap B_i) \cap B_i.$ Then $|R'| ≤ 1.5k \times 2k^2+k$.

Since $|S| ≤ 6(k−1)$ and none of the nodes of
$L$ is contained in $R$, therefore $|B - A - S - R - R' - X| \geq 2^{3k^2 + 4k - 1}$. In particular, each vertex in $B - A - S - R - R' - X$ has degree at least $6k$ in $G$ by (1). This implies that the average degree in $B - R$ is at least $6k - 1$. Hence by Mader’s theorem [32], there is a 1.5$k$-connected subgraph $W'_1$ in $B - X$. But then there must exist a separation $(A', B')$ of order at most $1.5k - 1$ in $G - X$ with $W'_1 \subseteq A'$, a contradiction. Note that the argument so far can be made in polynomial time, because all the proofs in Case 1.1 of Theorem 7.1 can be made in polynomial time. So if an odd $K_{2^{3k^2 + 4k}}$-model $L$ is given in $G$ by Theorem 7.1, we can make a reduction for $G$ in polynomial time, and hence we are done.

It remains to consider the case when $G$ does not contain an odd $K_{2^{3k^2 + 4k}}$-model. We need the following result in [10]

**Theorem 8.1.** Given a graph $G$ with no odd $K_t$-minor, there is an integer $f(t)$ with the following property; $V(G)$ can be decomposed into two parts $V_1, V_2$ with $V_1 \cup V_2 = V(G)$ such that both the graphs induced by $V_1$ and by $V_2$ are of tree-width at most $f(t)$. Moreover, such a partition can be found in polynomial time.

Let us apply Theorem 8.1 to $G - S$ with $t = 2^{3k^2 + 4k}$. Hence in polynomial time, $V(G - S)$ can be decomposed into two parts $V_1, V_2$ with $V_1 \cup V_2 = V(G - S)$ such that both the graphs induced by $V_1$ and by $V_2$ are of tree-width at most $f(t)$. By Theorem 3.2, in polynomial time, we can color the graphs $G_1$ and $G_2$ induced by $V_1$ and $V_2$, using at most $\chi(G)$ colors, respectively.

Hence in polynomial time, we can color $G = G_1 \cup G_2 \cup S$ using at most $2\chi(G) + 6(k - 1)$ colors. This completes the proof. ■