Approximate Counting via Correlation Decay on Planar Graphs

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Abstract
We show for a broad class of counting problems, correlation decay (strong spatial mixing) implies FPTAS on planar graphs. The framework for the counting problems considered by us is the Holant problems with arbitrary constant-size domain and symmetric constraint functions. We define a notion of regularity on the constraint functions, which covers a wide range of natural and important counting problems, including all multi-state spin systems, counting graph homomorphisms, counting weighted matchings or perfect matchings, and all counting CSPs and Holant problems with symmetric constraint functions of constant arity.

The core of our algorithm is a fixed-parameter tractable algorithm which computes the exact values of the Holant problems with regular constraint functions on graphs of bounded treewidth. By utilizing the locally tree-like property of apex-minor-free families of graphs, the parameterized exact algorithm implies an FPTAS for the Holant problem on these graph families whenever the Gibbs measure defined by the problem exhibits strong spatial mixing. We further extend the recursive coupling technique to establish the strong spatial mixing on Holant problems. As consequences, we have new deterministic approximation algorithms on planar graphs for several counting problems.

1 Introduction
In study of counting algorithms, many counting problems can be formulated as computing the partition function:

\[ Z(G(V, E)) = \sum_{\sigma \in [q]^V} \prod_{uv \in E} \Phi_E(\sigma(u), \sigma(v)) \prod_{v \in V} \Phi_V(\sigma(v)), \]

where \( \Phi_E : [q]^2 \to \mathbb{C} \) and \( \Phi_V : [q] \to \mathbb{C} \) are symmetric functions. This model is called spin system in Statistical Physics. It has vertices as variables and edges as constraints, and the partition function returns the total weight of all configurations. Many natural combinatorial problems such as counting independent sets, \( q \)-colorings, or graph homomorphisms can be expressed in this way.

We consider a framework that encompasses a much broader class of counting problems, namely, the Holant problems.

An instance of a Holant problem is an \( \Omega = (G(V, E), \{f_v\}_v \in V) \), where \( G \) is a graph, and each \( f_v \) is a function that maps tuples in \( [q]^{\deg(v)} \) to function values. The Holant of \( \Omega \) is defined as

\[ \text{hol}(\Omega) = \sum_{\sigma \in [q]^V} \prod_{v \in V} f_v(\sigma | E(v)), \]

where \( f_v(\sigma | E(v)) \) evaluates \( f_v \) on the restriction of \( \sigma \) on incident edges \( E(v) \) of vertex \( v \). The Holant problem \( \text{Holant}(G, F) \) specified by a graph family \( G \) and a function family \( F \), is the problem of computing \( \text{hol}(\Omega) \) for all valid instances \( \Omega \) defined by graphs from \( G \) and functions from \( F \).

The term Holant is coined by Valiant in [57] in studying of holographic algorithms. The formal framework of Holant problems is proposed in [11] by Cai, Lu and Xia.

The Holant framework is extremely expressive. Using the bipartite incidence graph to represent the participants of variables in constraints and choosing appropriate functions at vertices on both sides, computing the partition functions of spin systems and more generally counting CSPs can all be represented as special classes of Holant problems.

An algorithmic significance of Holant problems is that they are outcomes of holographic transformations. The holographic algorithms proposed by Valiant [56,57] compute exact solutions to the counting problems on planar graphs by transforming to problems solvable by the FKT algorithm [25, 38, 55] for counting planar perfect matchings. In the realm of approximate counting, perhaps the most successful (implicit) using of holographic transformation and Holant problem is the
FPRAS for ferromagnetic Ising model given by Jerrum and Sinclair in their seminal work [36]. The transformation in [36] from the spins world to the subgraphs world is indeed a holographic transformation, and the resulting subgraphs world problem is a Holant problem.\footnote{This actually happened more than a decade earlier than the concepts of holographic algorithm and Holant problem formally defined.} In these examples, the original counting problem is transformed to a Holant problem which has efficient exact or approximate algorithms. Therefore, the following problem is fundamental to the study of counting algorithms:

Characterize the tractability of exact computation and approximation of Holant\((\mathcal{G}, \mathcal{F})\) in terms of graph family \(\mathcal{G}\) and function family \(\mathcal{F}\).

**Exact computation.** The computation of exact value of Holant problem has been well studied on general graphs [7, 8, 10, 13–15, 43] and planar graphs [9, 12, 57], sometimes in form of dichotomy theorems, which states that every problem in the considered framework is either \#P-hard or having polynomial-time algorithm. Very recently, a dichotomy theorem [7] is proved for Holant problems with complex-valued functions on general graphs, concluding a long series of dichotomies on Holant problems. All these results consider Holant problems with boolean domain \((q = 2)\). Meanwhile, some special classes of Holant problems are more thoroughly understood, such as counting graph homomorphisms or counting CSP. For these specialized frameworks, dichotomy theorems were proved in a very general setting with complex-valued functions on generalized domains [6, 16]. See [18] for a good survey on these subjects.

Speaking very vaguely, the dichotomy theorems tell us that except for some rare cases almost all Holant problems are hard. Then a problem of algorithmic significance is to establish tractable results for Holant\((\mathcal{G}, \mathcal{F})\) on more refined graph families \(\mathcal{G}\), e.g. graphs with fixed parameters or forbidden minors, especially for general domain size \(q > 2\).

**Approximation.** We focus on deterministic approximate counting algorithms, specifically, the deterministic fully polynomial time approximation scheme (FPTAS). A central topic in this direction is the relation between correlation decay (strong spatial mixing) and approximability of counting.

Correlation decay is an important property of the marginal distribution, which is computationally equivalent to counting by the Jerrum-Valiant-Vazirani self-reduction [37]. The correlation decay property says that faraway vertices have little influence on the marginal distribution of local states, thus marginal probabilities should be well-approximated by local information only. However, as noted in [5, 33], this sufficiency of local information does not immediately yield efficient local computation. Two tools are invented to bridge this gap: the self-avoiding-walk tree (SAW-tree) of Weitz [59] and the computation tree of Gamarnik and Katz [33]. Both transform the original graph to a tree structure in which the marginal probabilities can be efficiently computed by recursions. With the SAW-tree the implication from strong spatial mixing to FPTAS is proved for 2-state spin systems [59], which becomes a foundation for several important algorithmic results [41, 42, 49, 51]. It is also proved in a long series of beautiful work [23, 31, 32, 47, 53, 54] that for the same class of counting problems lack of correlation decay implies inapproximability.

The relation between correlation decay and approximability for broader classes of counting problems is widely open, because of following technical challenges:

- It is known [52] that for domain size \(q > 2\) tree may not always represent the extremal case for correlation decay. Thus in order to use correlation decay to support approximate counting for those problems, the local computation has to be done on structures other than trees.
- Even on trees, the current recursion-based computation critically relies on the simplicity of constraint functions, as in the cases of spin systems and matchings. For general Holant problems, even on trees and when \(q = 2\), it is not known whether simple recursion exists.

**1.1 Our results.** We make progress on both exact and approximate computation of Holant problems by establishing connections between them.

We characterize a broad class of Holant problems whose exact computation is tractable on tree-like graphs and FPTAS is implied by strong spatial mixing on planar graphs. These Holant problems are characterized by a notion of regularity introduced by us on the constraint functions. Intuitively, being regular as a function means that the entropy of any input or partial input is constant. This covers a large family of important counting problems, including all spin systems, graph homomorphisms, counting CSP with symmetric constraints of bounded arity, matchings, perfect matchings, the subgraphs world problem in [36], etc.

For this broad class of counting problems, we give a fixed-parameter tractable algorithm which computes the exact value of counting in time \(2^{O(k)} \cdot \text{poly}(n)\) on graphs of size \(n\) and treewidth \(k\). Based on this parame-
parameterized algorithm, strong spatial mixing implies FPTAS on apex-minor-free graphs, which include planar graphs as special case.

We also apply the recursive coupling technique of Goldberg et al. [35] to analyze the strong spatial mixing for Holant problems. As examples, we have deterministic FPTAS on planar graphs, and more generally on all apex-minor-free graphs for the following counting problems:

- Counting $q$-colorings on triangle-free planar graphs of maximum degree $\Delta$ when $q > \alpha \Delta - \gamma$ where $\alpha \approx 1.76322$ and $\gamma \approx 0.47031$. This is just directly applying [35].

- The subgraphs world with parameter $\mu, \lambda < 1$ on planar graphs of maximum degree $\Delta$ when $\Delta < \frac{\ln(\frac{1}{\lambda})}{\ln(\frac{1}{\theta})}$, and as a consequence the ferromagnetic Ising model\(^2\) with inverse temperature $\beta$ and external field $B$ when $\Delta < \frac{1}{2} \left( \frac{\ln(\frac{1}{\lambda})}{\ln(\frac{1}{\theta})} \right)^2$.

- Ferromagnetic $q$-state Potts model of inverse temperature $\beta$ on planar graphs of maximum degree $\Delta$ when $\beta < \frac{\ln(q-1)}{\ln^2(q)}$, which vastly improves the mixing condition in [33] for FPTAS on general graphs and is close to the $\beta = O(\frac{1}{\Delta})$ bound conjectured in [33].

Technical contributions. Our parameterized algorithm does not directly use the tree decomposition. Instead, we define a new decomposition called the separator decomposition, which recursively separates the graph by small graph separators into components of limit-sized boundaries. This is quite different from the known treewidth-based approaches for spin systems, e.g. the junction tree algorithm; and this new construction more closely aligns with the conditional independence property: conditioning on any fixed assignment on a separator, the states of separated vertices are independent. The construction of separator decomposition makes explicit connections between the separable structure of tree-like graphs and the conditionally independent nature of counting problems defined by local constraints. As a result, our algorithm can deal with much broader class of counting problems other than just spin systems.

Unlike previous approximation algorithms via correlation decay, where the decay is verified on a tree of size exponential in the size of original graph, our FPTAS only relies on the correlation decay on the original graph. Thus we can directly apply those “decay-only” results such as [35] to get FPTAS. Since our approach does not explode the size of the graph, the FPTAS can even be supported by single-site correlation decays without requiring the amenability of graph as in [35].

1.2 Related work. The use of correlation decay technique for designing FPTAS for counting problems was initiated in [4, 59] and has been successfully applied to many problems [5, 31, 33], especially for computing the partition function of Ising model [41, 42, 51]. The technique of recursive coupling has been used to prove the property of correlation decay [34, 35, 44, 45].

The locally tree-like property of planar graphs and apex-minor-free graphs provides structure information to develop both exact and approximation algorithms on decision and optimization problems, some examples include [3, 21, 24, 27].

A framework for parameterized complexity of counting problems was proposed in [2, 28, 46]. The parameterized complexity of computing partition functions has been studied via probabilistic inference in graphical model [17]. Some logical approaches have also been extended to counting problems on structures with small treewidth or local treewidth [1, 19, 30].

2 Models and statement of results

2.1 Holant problems. Let $[q] = \{0, 1, \ldots, q - 1\}$ be a domain of size $q$, where $q \geq 2$ is an finite integer.

Let $f : [q]^d \to \mathbb{F}$ be a $d$-ary function where $\mathbb{F}$ is a field. In this paper, we consider either $\mathbb{F} = \mathbb{C}$ the complex and $\mathbb{F} = \mathbb{R}^+$ the nonnegative reals. To avoid issues of computation model, we assume all number are algebraic. We allow the function arity $d$ to be $0$. When $d = 0$, the only member of $[q]^0$ is the empty tuple $\varepsilon$, and a $0$-ary function $f$ maps $\varepsilon$ to a function value. We call such function $f$ a trivial function.

A $d$-ary function is symmetric if $f(x_1, \ldots, x_d) = f(x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ for any permutation $\sigma$ of $\{1, 2, \ldots, d\}$. When $q = 2$, functions have boolean domain and a $d$-ary symmetric function $f$ can be denoted by $[f_0, f_1, \ldots, f_d]$ where $f_k$ specifies the function value for the input tuple with Hamming weight $k$. For example the equality function is denoted as $1, 0, 0, \ldots, 0, 1$.

Let $\Phi_E : [q]^2 \to \mathbb{C}$ and $\Phi_V : [q] \to \mathbb{C}$ be two symmetric functions. The partition function of an undirected graph $G(V, E)$ is defined as

$$Z(G) = \sum_{\sigma \in [q]^V} \prod_{\{u, v\} \in E} \Phi_E(\sigma(u), \sigma(v)) \prod_{v \in V} \Phi_V(\sigma(v)).$$

This is called a $q$-state spin system.
(G(V, E), \{f_v\}_{v \in V}) be an instance, where each f_v, called a constraint function or a signature, is a d-ary symmetric function with d = \deg(v). We define the Holant of \( \Omega \) as
\[
\text{hol}(\Omega) = \sum_{\sigma \in [q]^d} \prod_{v \in V} f_v(\sigma | E(v)),
\]
where \( f_v(\sigma | E(v)) \) evaluates \( f_v \) on the restriction of \( \sigma \) on incident edge of \( v \).

Let \( \mathcal{G} \) be a family of graphs and \( \mathcal{F} \) be a family of functions. A Holant problem Holant(\( \mathcal{G}, \mathcal{F} \)) is a computation problem that given as input an instance \( \Omega = (G(V, E), \{f_v\}_{v \in V}) \) where \( G \in \mathcal{G} \) and all \( f_v \) are from \( \mathcal{F} \), compute hol(\( \Omega \)).

A symmetric function can be represented by a vector enumerating the function values for all inputs (up to symmetry). The number of symmetry classes of \( \sigma \in [q]^d \) equals the number of weak \( q \)-composition of integer \( d \), which is \( \binom{d+q-1}{q-1} \). Thus symmetric \( d \)-ary functions can be represented by vectors of length polynomial in \( d \).

Spin systems can be represented as special class of Holant problems. For a graph \( G(V, E) \), let \( \mathcal{I}_G \) denote the incidence graph of \( G \), i.e. \( \mathcal{I}_G = (V_1, V_2, E') \) is a bipartite graph with \( V_1 = V, V_2 = E \) and \( (v, e) \in E' \) if and only if edge \( e \) is incident to vertex \( v \) in \( G \).

For a spin system defined by functions \( \Phi_E \) and \( \Phi_V \) on a graph \( G \), we can transform it to a Holant instance \( \Omega = (\mathcal{I}_G, \{f_v\}_{v \in V \cup E}) \), where \( f_v = \Phi_E \) for right vertices \( v \in E \) and for left vertices \( v \in V \), \( f_v \) is the generalized Equality function defined as \( f(x_1, \ldots, x_d) = \Phi_V(x_1) \) if \( x_1 = \cdots = x_d \) and \( f(x_1, \ldots, x_d) = 0 \) otherwise. It is easy to check that \( \text{hol}(\Omega) = Z(G) \).

### 2.2 Regular functions
We introduce a notion of regularity of constraint functions, which is characterized by the “pinning” operation on symmetric functions. The pinning operation on a function defines a new function with smaller arity by fixing (pinning) the values of some of the variables.

**Definition 2.1. (Pinning)** Let \( f : [q]^d \to \mathbb{F} \) be a d-ary symmetric function. Let \( 0 \leq k \leq d \) and \( \tau \in [q]^k \). We define that \( \text{PIN}_\tau(f) = g \) where \( g : [q]^{d-k} \to \mathbb{F} \) is a \((d-k)\)-ary symmetric function such that \( \forall \sigma \in [q]^{d-k}, \)
\[
g(\sigma) = f(\sigma(1), \ldots, \sigma(d-k), \tau(1), \ldots, \tau(k)).
\]
Specifically, when \( k = 0 \) the resulting function \( g = f \); and when \( k = d \), the resulting function \( g \) is a trivial function \( f(\sigma) \).

Note that since \( f \) is symmetric, the positions of \( \tau(1), \ldots, \tau(k) \) in \( f \) does not matter, and the pinning of a symmetric function is still symmetric.

To exemplify the effect of pinning, consider the case when \( q = 2 \) and a function \( f \) is represented in form \( [f_0, f_1, \ldots, f_d] \). For a \( \sigma \in [2]^k \) that \( \sigma \) has \( t \) many 1s, we have \( \text{PIN}_\tau(f) = [f_0, f_1, \ldots, f_{d-t}] \). That is, the \( \text{PIN}_\tau(f) \) for a \( \sigma \in [2]^k \) returns a “sliding window” of length \( d-k \) in \([f_0, f_1, \ldots, f_d]\) whose starting position is determined by the number of 1s in \( \sigma \).

A notion of regularity of symmetric functions can be defined by limiting the outcomes of pinning.

**Definition 2.2. (Regularity)** A symmetric function \( f : [q]^d \to \mathbb{F} \) is called C-regular if for all \( 0 \leq k \leq d \), it holds that
\[
\left| \left\{ \text{PIN}_\tau(f) \mid \forall \tau \in [q]^k \right\} \right| \leq C.
\]
A family \( \mathcal{F} \) of symmetric functions is called regular if there exists a finite constant \( C > 0 \) such that every \( f \in \mathcal{F} \) is C-regular.

The following sufficient conditions for regular symmetric functions can be easily verified.

**Proposition 2.1.** Let \( f : [q]^d \to \mathbb{F} \) be a symmetric function. For \( \sigma \in [q]^d \) and \( i \in [q] \), let \( n_i(\sigma) = |\{1 \leq j \leq d \mid \sigma(j) = i\}| \) be the number of \( i \)-entries in \( \sigma \).

- (bounded arity) \( f \) is \((d+q-1)\)-regular.
- (cyclic) If there is a \( c > 0 \) such that \( f(\sigma) \) depends only on \( (n_1(\sigma) \mod c, \ldots, n_q(\sigma) \mod c) \) then \( f \) is \( c^q-1 \)-regular.
- (constant exceptions) If there is a C-regular \( g : [q]^d \to \mathbb{F} \) and a \( c \geq 0 \) such that \( f \equiv g \) differ only at those \( \sigma \in [q]^d \) that \( n_i(\sigma) \geq d-c \) for some \( i \in [q] \), then \( f \) is \( (C + q \cdot (c^q-1)) \)-regular.

As consequences, all constant-ary symmetric functions, Equality and generalized Equality, and the Not-All-Equal are all regular. This covers all q-state spin systems.

For boolean domain, a function \([f_0, f_1, \ldots, f_d] \) is regular either if it is cyclic, i.e. \( f_k = \lambda_k \mod c \) for some constant \( c \), or it becomes cyclic after removing constant many exceptions \( f_0, f_1, \ldots, f_{d-c}, \ldots, f_d \) from both ends. This covers (weighted) matchings, perfect matchings, the subgraphs world transformed from the Ising model [36], which is a Holant problem defined by constraint functions in the form \([1, \mu, 1, \ldots] \) and \([1, 0, \lambda] \).

### 2.3 Correlation decay
Consider a Holant instance \( \Omega = (G(V, E), \{f_v\}_{v \in V}) \) where each \( f_v : [q]^\deg(v) \to \mathbb{R}^+ \) is a symmetric function with nonnegative real function...
values. Each \( \sigma \in [q]^E \) is called a configuration and 
\[ w(\sigma) = \prod_{v \in V} f_v(\sigma | \sigma_{\partial(v)}) \] is its weight.

A configuration \( \sigma \in [q]^E \) is feasible if \( w(\sigma) > 0 \).
And for a configuration \( \tau_A \in [q]^A \) on a subset \( A \subseteq E \) of edges, we say that \( \tau_A \) is feasible if there is a feasible \( \sigma \in [q]^E \) agreeing with \( \tau_A \) on \( A \).

The Gibbs measure is a probability distribution over all configurations, defined as \( \mu(\sigma) = \frac{w(\sigma)}{\max(\{w(\sigma) : \sigma \in [q]^E\})} \). To make the Gibbs measure well-defined, we require that each \( f_v \) has nonnegative values and the Holant problem is feasible, i.e., there exists a feasible configuration.

For a feasible \( \sigma_A \in [q]^A \) on \( A \subseteq E \), we use \( \mu^{\sigma_A} \) to denote the marginal distribution at \( e \) conditioning on the configuration of \( \Lambda \) being fixed as \( \sigma_\Lambda \).

**Definition 2.3. (Strong Spatial Mixing)** A Holant problem \( \text{Holant}(G, F) \) has strong spatial mixing (SSM) if for any instance \( \Omega = (G(V, E), \{f_v\}_{v \in V}) \), any \( e \in E \), \( \Lambda \subseteq E \) and any two feasible configurations \( \sigma_A, \tau_A \in [q]^A \), it holds that

\[ ||\mu^{\sigma_A} - \mu^{\tau_A}||_{TV} \leq \text{Poly}(|V|) \cdot \exp(-\Omega(\text{dist}(e, \Delta))), \]

where \( \Delta \subseteq \Lambda \) is the subset on which \( \sigma_\Lambda \) and \( \tau_\Lambda \) differ, \( \text{dist}(e, \Delta) \) is the shortest distance from edge \( e \) to any edges in \( \Delta \), and \( || \cdot ||_{TV} \) denotes the total variation distance.

Our notion of SSM is a generalization of SSM for spin systems of Weitz [59] to Holant problems, and is different from the SSM used in [35] where \( \Delta \) contains only one edge. We call the latter one single-site strong spatial mixing (SSSSM).

**2.4 Tractable search.** In order to apply the self-reduction technique of Jerrum-Valiant-Vazirani [37] for approximate counting, we also require that the following search problem is tractable:

**Input:** a Holant instance \( \Omega = (G(V, E), \{f_v\}_{v \in V}) \), and a configuration \( \sigma_A \in [q]^A \) on \( A \subseteq E \);

**Output:** a feasible \( \tau \in [q]^E \) agreeing with \( \sigma_A \) on \( A \), or determines no such \( \tau \) exists.

We call such property the tractable search for \( \text{Holant}(G, F) \). We remark that this is a very natural assumption for approximate counting: for all known examples of approximate counting implied by mixing, the above search problem is easy or even trivial. The tractable search requirement of the general Holant framework is an analog to the specific \( q \geq \Delta + 1 \) requirement for counting \( q \)-coloring.

The tractable search is related to the Holant\(^c\) framework. For \( i \in [q] \), let \( \Delta_i \) denote the unary function which maps \( i \) to 1 and all other \( j \in [q] \) to 0.

**Definition 2.4.** \( \text{Holant}^c(G, F) = \text{Holant}(G, F \cup \{\Delta_i \mid i \in [q]\}) \).

The Holant\(^c\) problem has significance in complexity of counting [8, 13]. Assuming the tractable search for \( \text{Holant}(G, F) \) is equivalent to assuming the polynomial-time decision oracle (for existence of feasible configuration) for \( \text{Holant}^c(G, F) \).

**2.5 Local treewidth and planarity.** Our algorithm relies on the treewidth of graph and family of graphs with forbidden graph minors. We will not formally define these concepts excepting saying that the treewidth of a graph \( G \), denoted by \( tw(G) \), measures how similar the graph \( G \) is to a tree. The formal definitions can be found in standard textbooks, e.g. [22].

Graphs of bounded local treewidth are precisely the family of apex-minor-free graphs, where an apex graph has a vertex whose removal leaves a planar graph. In particular, \( K_5 \) and \( K_{3,3} \) are apex graphs, therefore apex-minor-free graphs include planar graphs as a special case.

**Theorem 2.1.** ([20, 24]) Let \( G \) be an apex-minor-free family of graphs. For any \( G(V, E) \in G \) and \( v \in V \), let \( N_r(v) \) be the subgraph of \( G \) induced by vertices whose distance to \( v \) is at most \( r \). Then \( tw(N_r(v)) \leq f(r) \) for some linear function \( f \).

The following easy proposition on treewidth of incident graphs implies that representing spin systems as Holant problem on the incidence graphs does not violate the graph structure.

**Proposition 2.2.** Let \( G \) be a graph and \( I_G \) be the bipartite incidence graph of \( G \). Then \( tw(G) = tw(I_G) \).

**2.6 Main results.** Our main results can be summarized by the following two theorems.

**Theorem 2.2.** There is an algorithm for \( \text{Holant}(G, F) \) with regular symmetric \( F \), whose running time is \( 2^{O(k)} \cdot \text{poly}(n) \) for any \( G \in G \) of \( n \) vertices and treewidth \( k \).

**Theorem 2.3.** Assuming the tractable search, for Holant problem \( \text{Holant}(G, F) \) with apex-minor-free \( G \) and regular symmetric nonnegative \( F \), SSM implies existence of FPTAS.

We further stress that other than the apex-minor-free-ness, Theorem 2.3 does not rely on any additional assumption on graph structure, e.g. bounded growth or amenability which were used in [35, 58].

Applying Theorem 2.3 with correlation decay results, we have FPTAS for several counting problems, which are stated in Section 6.4.

**3 Structure of regular functions**

The main purpose of this section is to prove some useful properties of regular symmetric functions, namely,
Lemma 3.1, 3.3 and 3.4, which are all essential to our counting algorithms. Towards this goal, some new definitions are introduced and some new lemmas are proved.

Note that different \( \sigma, \tau \in [q]^k \) (up to symmetry) may yield the same function after pinning \( f \) with them. We classify members of \([q]^k\) into equivalence classes according to their effects of pinning on \( f \) by introducing the following concept of peers.

**Definition 3.1. (Peers)** Let \( f : [q]^d \rightarrow \mathbb{F} \) be a d-ary symmetric function. Let \( 0 \leq k \leq d \) and \( \tau \in [q]^k \). We define that \( \text{Peer}_\tau(f) = g \) where \( g : [q]^k \rightarrow \{0,1\} \) is a boolean symmetric function such that

\[
\forall \sigma \in [q]^k, \quad g(\sigma) = \begin{cases} 1 & \text{if } \text{Pin}_\tau(f) = \text{Pin}_\tau(f), \\ 0 & \text{otherwise}. \end{cases}
\]

We also interpret \( \text{Peer}_\tau(f) \) as a set and write \( \sigma \in \text{Peer}_\tau(f) \) if \( \text{Peer}_\tau(f)(\sigma) = 1 \).

The condition \( \text{Pin}_\sigma(f) = \text{Pin}_\tau(f) \) defines an equivalence relation between \( \sigma \) and \( \tau \) by requiring them having the same effect of pinning on \( f \). Then \( \text{Peer}_\tau(f) \) is the indicator function of the equivalence class \( \{ \sigma \in [q]^k \mid \text{Pin}_\sigma(f) = \text{Pin}_\tau(f) \} \). So we have the following easy but useful proposition.

**Proposition 3.1.** It holds that \( \sigma \in \text{Peer}_\tau(f) \) if and only if \( \text{Peer}_\sigma(f) = \text{Peer}_\tau(f) \).

The peer images of an input uniquely determines the function value, specifically:

**Lemma 3.1.** Let \( f : [q]^d \rightarrow \mathbb{F} \) be a symmetric function. Let \( r \geq 1, d_1 + d_2 + \cdots + d_r = d \), and \( \sigma_1, \tau_1 \in [q]^{d_1} \) for \( i = 1, 2, \ldots, r \). If \( \text{Peer}_{\sigma_i}(f) = \text{Peer}_{\tau_i}(f) \) for all \( i = 1, 2, \ldots, r \), then \( f(\tau_1 \tau_2 \cdots \tau_r) = f(\sigma_1 \sigma_2 \cdots \sigma_r) \).

**Proof.** Due to Proposition 3.1, \( \tau_i \in \text{Peer}_{\sigma_i}(f) \) for all \( i = 1, 2, \ldots, r \). We prove by induction on \( r \). When \( r = 1 \), for all \( d \geq 0 \) and \( \sigma_1 \in [q]^d \), \( \text{Pin}_{\tau_1}(f) \) is a trivial function \( f(\sigma_1) \) (a function value) and \( \text{Peer}_{\tau_1}(f) \) is the equivalence class of all \( \sigma_1 \in [q]^d \) which have the same \( \text{Pin}_{\tau_1}(f) \) as \( \text{Pin}_{\sigma_1}(f) \), i.e., \( f(\tau_1) = f(\sigma_1) \).

Assume the statement holds for all smaller \( r \) and all \( d \). Let \( \sigma_1, \tau_1 \in [q]^{d_i}, i = 1, 2, \ldots, r \) satisfy that \( \tau_i \in \text{Peer}_{\sigma_i}(f) \) for all \( i = 1, 2, \ldots, r \). Since \( \tau_i \in \text{Peer}_{\sigma_i}(f) \), we have \( \text{Pin}_{\tau_i}(f) = \text{Pin}_{\sigma_i}(f) \). Denote that \( g = \text{Pin}_{\tau_i}(f) = \text{Pin}_{\sigma_i}(f) \). By definition of pinning, it holds that \( f(\sigma_1 \sigma_2 \cdots \sigma_r) = g(\sigma_1 \sigma_2 \cdots \sigma_{r-1}) \) and \( f(\tau_1 \tau_2 \cdots \tau_r) = g(\tau_1 \tau_2 \cdots \tau_{r-1}) \). Note that \( g \) satisfies the induction hypothesis for \( r-1 \), which means that \( g(\tau_1 \tau_2 \cdots \tau_{r-1}) = g(\sigma_1 \sigma_2 \cdots \sigma_{r-1}) \) as long as \( \tau_i \in \text{Peer}_{\sigma_i}(f) \) for all \( i = 1, 2, \ldots, r-1 \). Therefore, for \( \sigma_1, \tau_1 \in [q]^{d_i}, i = 1, 2, \ldots, r \) that \( \tau_i \in \text{Peer}_{\sigma_i}(f) \) for all \( i = 1, 2, \ldots, r \), we have \( f(\tau_1 \tau_2 \cdots \tau_r) = g(\tau_1 \tau_2 \cdots \tau_{r-1}) = g(\sigma_1 \sigma_2 \cdots \sigma_{r-1}) = f(\sigma_1 \sigma_2 \cdots \sigma_r) \).

Note that the outcome of \( \text{Peer}_\tau(f) \) is still a symmetric function, so we can apply pinning and peering operations on it. We can define the peering closure which contains all possible outcomes of recursively applying peer operations on a function \( f \).

**Definition 3.2. (Peering closure)** Let \( f : [q]^d \rightarrow \mathbb{F} \) be a d-ary symmetric function. Let \( 0 \leq k_r \leq k_{r-1} \leq \cdots \leq k_1 \leq d \) and \( \tau_i \in [q]^{k_i}, 1 \leq i \leq r \). We denote that

\[
\text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) = \text{Peer}_{\tau_r}(\text{Peer}_{\tau_1, \tau_2, \ldots, \tau_{r-1}}(f)).
\]

The peering closure of \( f \), denoted by \( \text{Peer}^*(f) \), is defined as

\[
\text{Peer}^*(f) = \{ \text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) \mid \tau_i \in [q]^{k_i} \text{ for } 1 \leq i \leq r, \\
0 \leq k_r \leq \cdots \leq k_1 \leq d, r \geq 1 \}.
\]

Note that \( \text{Peer}_\tau(f) \) is a boolean function no matter what the range of \( f \) is. A boolean function \( g : [q]^d \rightarrow \{0,1\} \) can be seen equivalently as a set \( \{ \sigma \in [q]^d | g(\sigma) = 1 \} \). For two boolean functions \( g \) and \( h \) defined on the same domain \([q]^d\), we define the operations on boolean functions \( g \cup h, g \cap h, \) and \( g \subseteq h \) according to the operations on their set representations.

Some useful properties of peering are better presented in this set language:

**Lemma 3.2.** Let \( f : [q]^d \rightarrow \mathbb{F} \) be a symmetric function, and \( g, h : [q]^d \rightarrow \{0,1\} \) be boolean symmetric functions. Let \( \sigma \in [q]^d \) and \( \tau \in [q]^k \) for arbitrary \( 0 \leq k \leq \ell \leq d \). We have

1. \( \text{Peer}_\tau(f) \subseteq \text{Peer}_\tau(\text{Peer}_\sigma(f)) ; \\
2. (\text{Peer}_\tau(g) \cap \text{Peer}_\tau(h)) \subseteq \text{Peer}_\tau(g \cup h) \).

**Proof.** Both statements can be proved by directly expanding the definition of \( \text{Peer}(-) \).

1. For any \( \pi \in [q]^k \), suppose that \( \pi \in \text{Peer}_\tau(f) \), which means \( \text{Pin}_\tau(f) = \text{Pin}_\pi(f) \). Then for any \( x \in [q]^d \), we have \( \text{Pin}_{x \pi}(f) = \text{Pin}_{x \tau}(f) \) and only if \( \text{Pin}_{x \tau}(f) = \text{Pin}_{\pi}(f) \), which is equivalent to that for any \( x \in [q]^d \), \( x \tau \in \text{Peer}_\pi(f) \) and only if \( x \tau \in \text{Peer}_\tau(f) \). This implies that \( \text{Pin}_{\pi}(\text{Peer}_\tau(f)) = \text{Pin}_{\tau}(\text{Peer}_\pi(f)) \), which implies \( \pi \in \text{Peer}_\tau(\text{Peer}_\pi(f)) \). Therefore, \( \text{Peer}_\tau(f) \subseteq \text{Peer}_\tau(\text{Peer}_\pi(f)) \).

2. For any \( \pi \in [q]^k \), suppose that \( \pi \in \text{Peer}_\tau(g) \cap \text{Peer}_\tau(h) \), which implies that \( \text{Pin}_\tau(g) = \text{Pin}_\pi(g) \) and \( \text{Pin}_\tau(h) = \text{Pin}_\pi(h) \). Then for any \( x \in [q]^d \),
it holds that $x \tau \in g$ if and only if $x \pi \in g$ and $x \tau \in h$ if and only if $x \pi \in h$, thus $x \tau \in g \cup h$ if and only if $x \pi \in g \cup h$, which means $\text{Pin}_r(g \cup h) = \text{Pin}_r(g \cup h)$, thus $\pi \in \text{Peer}_r(g \cup h)$. Therefore, $(\text{Peer}_r(g) \cap \text{Peer}_r(h)) \subseteq \text{Peer}_r(g \cup h)$.

We then characterize all $k$-ary functions in peering closure by unions of boolean functions.

**Lemma 3.3.** Let $f : [q]^d \to \mathbb{F}$ be a symmetric function. For any $0 \leq k \leq d$, every $k$-ary function $g \in \text{Peer}^*(f)$ can be represented as

$$g = \bigcup_{i=1}^t \text{Peer}_{\sigma_i}(f)$$

for some $\sigma_1, \ldots, \sigma_t \in [q]^k$.

**Proof.** Recall that every $k$-ary function in $\text{Peer}^*(f)$ is in the following form:

$$\text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) = \text{Peer}_{\tau_r} \left( \left( \cdots \left( \text{Peer}_{\tau_{r-1}}(f) \right) \right) \cdots \right),$$

for some $r \geq 1$, $k = k_r \leq k_{r-1} \leq \cdots \leq k_1 \leq d$, and $\tau_i \in [q]^{k_i}$, $1 \leq i \leq r$.

We then prove by induction on $r$ that

$$\text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) = \text{Peer}_{\sigma_1}(f) \cup \cdots \cup \text{Peer}_{\sigma_t}(f)$$

for some $\sigma_1, \ldots, \sigma_t \in [q]^k$, where $k$ is the arity of $\tau_r$.

When $r = 1$, this is trivially true. Assume the statement holds for all smaller $r$. Then $\text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) = \text{Peer}_{\tau_r} \left( \left( \cdots \left( \text{Peer}_{\tau_{r-1}}(f) \right) \right) \cdots \right)$. And due to the induction hypothesis, there exist $\sigma_1, \ldots, \sigma_t \in [q]^{k_{r-1}}$ such that

$$\text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) = \bigcup_{i=1}^t \text{Peer}_{\sigma_i}(f).$$

Due to Lemma 3.2, we have that $\text{Peer}_{\tau_r}(f) \subseteq \text{Peer}_{\tau_r}(\text{Peer}_{\sigma}(f))$ for any $\sigma \in [q]^{k_{r-1}}$ and

$$\bigcup_{i=1}^t \text{Peer}_{\tau_r}(\text{Peer}_{\sigma_i}(f)) \subseteq \text{Peer}_{\tau_r} \left( \bigcup_{i=1}^t \text{Peer}_{\sigma_i}(f) \right).$$

Combining these, we have

$$\text{Peer}_{\tau_r}(f) \subseteq \text{Peer}_{\tau_r} \left( \bigcup_{i=1}^t \text{Peer}_{\sigma_i}(f) \right)$$

$$= \text{Peer}_{\tau_r} \left( \text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) \right)$$

$$= \text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f).$$

Note that this already implies the lemma, i.e. there exist $t \geq 1$ and $\sigma_1, \ldots, \sigma_t \in [q]^k$ where $k$ is the arity of $\tau_r$ such that $\text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) = \bigcup_{i=1}^t \text{Peer}_{\sigma_i}(f)$. To see this, by contradiction we assume that the statement is false. Then there must exist $\sigma_1, \sigma_2 \in [q]^k$ such that $\sigma_1, \sigma_2 \notin \text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f)$ but $\sigma_1 \in \text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f)$ and $\sigma_2 \notin \text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f)$. Recall that $\text{Peer}_{\sigma}(f)$ are equivalent classes. Then the condition

$$\sigma_1 \in \text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) = \text{Peer}_{\tau_r} \left( \text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) \right)$$

implies that

$$\text{Peer}_{\sigma_1} \left( \text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) \right)$$

$$= \text{Peer}_{\tau_r} \left( \text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) \right).$$

On the other hand, we already show that

$$\text{Peer}_{\sigma_1}(f) \subseteq \text{Peer}_{\sigma_1} \left( \text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) \right).$$

However, due to the assumption we have

$$\sigma_2 \notin \text{Peer}_{\tau_1, \tau_2, \ldots, \tau_r}(f) = \text{Peer}_{\sigma_1} \left( \text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) \right),$$

which implies that

$$\text{Peer}_{\sigma_1}(f) \notin \text{Peer}_{\sigma_1} \left( \text{Peer}_{\tau_1, \ldots, \tau_{r-1}}(f) \right),$$

a contradiction.

**Lemma 3.4.** Let $f : [q]^d \to \mathbb{F}$ be a symmetric function. If $f$ is $C$-regular, then

1. for any $0 \leq k \leq d$, the number of distinct $k$-ary functions in $\text{Peer}^*(f)$ is at most $2^C$;

2. for every $g \in \text{Peer}^*(f)$, $g$ is $C$-regular.

**Proof.** First note that for any $0 \leq k \leq d$, it holds that

$$\left| \left\{ \text{Peer}_\tau(f) \mid \tau \in [q]^k \right\} \right| = \left| \left\{ \text{Pin}_\tau(f) \mid \tau \in [q]^k \right\} \right|.$$ 

This is because they both count the number of equivalence classes defined by the equivalence relation that $\sigma$ and $\tau$ are equivalent if and only if $\text{Pin}_\sigma(f) = \text{Pin}_\tau(f)$. If $f$ is $C$-regular, then $\left| \left\{ \text{Pin}_\tau(f) \mid \tau \in [q]^k \right\} \right| \leq C$, so we have $\left| \left\{ \text{Peer}_\tau(f) \mid \tau \in [q]^k \right\} \right| \leq C$.

Due to Lemma 3.3, every $k$-ary function in $\text{Peer}^*(f)$ can be represented as a (boolean-functional) union $\bigcup_{i=1}^t \text{Peer}_{\sigma_i}(f)$ for some $\sigma_1, \ldots, \sigma_t \in [q]^k$. The number of different unions is obviously bounded by the size of power set of $\left\{ \text{Peer}_\tau(f) \mid \tau \in [q]^k \right\}$, which is at most $2^C$. Therefore the number of distinct $k$-ary functions in $\text{Peer}^*(f)$ is at most $2^C$. The first part of the lemma is proved.
Let $0 \leq \ell \leq k \leq d$ and $\tau \in [q]^{k}$. By definition of \textsc{Peer}$(\cdot)$, both $\{\text{Peer}_\tau(f)\mid \sigma \in [q]^{\ell}\}$ and $\{\text{Peer}_\sigma(\text{Peer}_\tau(f))\mid \sigma \in [q]^{k}\}$ are partitions of $[q]^{\ell}$. Due to Lemma 3.2, for any $(a$ leaf) with each node of $G$
\begin{equation}
\text{Definition 4.2. (Separator Decomposition)}
\end{equation}Note that in the above definition we do not require $X$ be a subgraph of $G$.
\begin{equation}
\text{Theorem 4.2. (Reed [48], Robertson-Seymour [50])}
\end{equation}Let $G = (V,E)$ be an undirected graph. If $\text{tw}(G) \leq k$,
then for every $W \subseteq V$ of size at least $2k + 3$ there exists a balanced $W$-separator of size at most $k + 1$. Conversely, if for every $W \subseteq V$ of size $3k + 1$ there exists a balanced $W$-separator of size at most $k + 1$, then $tw(G) \leq 4k + 1$.

We then use balanced separators to characterize the width of separator decompositions.

**Lemma 4.1.** Let $G = (V, E)$ be a graph of $n$ vertices.

1. If $G$ has a separator decomposition of width $s$, then for every $W \subseteq V$ of size at least $6s$ there is a balanced $W$-separator of size at most $2s$.

2. If for every $W \subseteq V$ of size $6s$ there is a balanced $W$-separator of size at most $2s$, then $G$ contains a separator decomposition $T$ of width at most $6s$. And such $T$ can be constructed in time $2^{O(s)} \cdot \text{poly}(n)$.

**Proof.** We show the first part: existence of a separator decomposition of width $s$ implies that for every $W \subseteq V$ there is a balanced $W$-separator of size at most $2s$.

Let $T$ be a separator decomposition of $G$ of width $s$, with each node $i \in T$ associated with a vertex set $V_i$ and a separator $S_i$ of $G[V_i]$. It holds that $\partial V_i \leq s$ and $|S_i| \leq s$ for all $i \in T$.

Fix an arbitrary $W \subseteq V$ with $|W| \geq 6s$. Let $i$ be the node in $T$ of maximum depth satisfying $|V_i \cap W| > \frac{1}{2}|W|$. Such node $i$ always exists and must be a non-leaf since $V_i \cap W = W$ for the root $r$ and $V_i \cap W = \emptyset$ for every leaf $\ell$. Let $S = \partial V_i \cup S_i$. It holds that $|S| \leq |\partial V_i| + |S_i| \leq 2s$. We then show that $S$ is a balanced $W$-separator.

Let $j, k$ be the children of $i$ in $T$. Due to the maximality of the depth of node $i$, it holds that $|V_j \cap W| \leq \frac{1}{2}|W|$ and $|V_k \cap W| \leq \frac{1}{2}|W|$. We distinguish between the following two cases:

- Case 1: $|V_j \cap W| < \frac{1}{2}|W|$ and $|V_k \cap W| < \frac{1}{2}|W|$. Let $X = (V_j \cap W) \cup (V_k \cap W)$ and $Y = W \setminus (X \cup S)$. We have $|X| \leq |V_j \cap W| + |V_k \cap W| < \frac{3}{4}|W|$ and $|Y| = |W \setminus (X \cup S)| \leq |W| - |V_j \cap W| \leq \frac{1}{2}|W| < \frac{3}{4}|W|$.

- Case 2: $|V_j \cap W| \geq \frac{1}{2}|W|$ or $|V_k \cap W| \geq \frac{1}{2}|W|$. Without loss of generality, suppose that $|V_j \cap W| \geq |V_k \cap W|$ and $|V_j \cap W| \geq \frac{1}{2}|W|$. Let $X = V_j \cap W$, $Y = W \setminus (S \cup X)$. We have that $|X| = |V_j \cap W| \leq \frac{1}{2}|W|$ and $|Y| \leq |W| - |V_j \cap W| \leq \frac{1}{2}|W|$. Also $X$ and $Y$ are nonempty since $|X| = |V_j \cap W| \geq \frac{1}{2}|W| > 0$ and $|Y| \geq |W| - |X| - |S| > \frac{3}{4}|W| > 3s - 2s > 0$.

We then show that $X$ and $Y$ are separated by $S$ in $G$. It holds that $X \subseteq V_j$ and $Y \cap V_j = \emptyset$, thus every path $X$ to $Y$ must go through some vertex in $\partial V_i \subseteq S$.

Therefore, in both cases $S$ is a balanced $W$-separator. The first part of the lemma is proved.

We then prove the second part: if for every $W \subseteq V$ of size $6s$ there is a balanced $W$-separator of size at most $2s$, then $G$ contains a separator decomposition of width at most $6s$. We first prove the following claim.

**Claim 4.1.** If for every $W \subseteq V$ of size $6s$ there is a balanced $W$-separator of size at most $2s$, then for any nonempty $R \subseteq V$ with $|R| \leq 6s$, there is a partition $\{X, Y, S\}$ of $R$ such that

1. $S$ is an $(X, Y)$-separator in $G[R]$ and $|S| \leq 4s$;
2. $|\partial X|, |\partial Y| \leq 6s$.

**Proof.** When $|R| \leq 4s$, the claim holds by taking $S = R$ and $X = Y = \emptyset$. We then consider only the case $|R| > 4s$.

Let $W \supseteq \partial R$ and $|W| = 6s$. Let $S'$ be a balanced $W$-separator in $G$ of size at most $2s$. Then $W \setminus S'$ can be partitioned into $X_V$ and $Y_V$ such that $X_V$ and $Y_V$ are disconnected in $G[V \setminus S']$, and $0 < |X_V|, |Y_V| \leq \frac{3}{4}|W|$. Since $G[V \setminus S']$ is disconnected, we have that $V \setminus S'$ can be partitioned into $X_V$ and $Y_V$ such that $X_V \subseteq X_V$, $Y_V \subseteq Y_V$, $X_V$ and $Y_V$ are disconnected in $G[V \setminus S']$, and $0 < |X_V|, |Y_V| \leq \frac{3}{4}|W|$. We define that $S = S' \cap R$, $X = X_V \cap R$ and $Y = Y_V \cap R$. We then verify that they satisfy the requirement.

If $X = R$ (or $Y = R$) then $S = \emptyset$, and $X_V$ (respectively $Y_V$) contains more than $4s$ vertices in $W$, contradicting that $S'$ is a balanced $W$-separator. And it follows from that $S'$ separates $X_V$ and $Y_V$ in $G$ that $S$ is an $(X, Y)$-separator in $G[R]$. It trivially holds that $|S| \leq |S'| \leq 2s$.

It is easy to see that $\partial X \subseteq X_W \cup S'$, therefore $|\partial X| \leq |X_W| + |S'| \leq 6s$. The same holds for $\partial Y$.

Applying the above claim we can construct the separator decomposition $T$ for a graph $G(V, E)$ as follows. Initially, for the root $r$ of $T$, let $V_r = V$. For any current node $i \in T$, if $V_i \neq \emptyset$, set $R = V_i$ and apply the above
claim to get an \((X, Y)\)-separator of \(S\) in \(G[R]\) with desirable properties. Then create two children \(i\) and \(j\) in \(T\), let \(V_i = X\), \(V_j = Y\), and recursively do the same thing for the two children.

It is easy to see that \(T\) is a separator decomposition for \(G\) of width at most 6s. For \(|W| = 6s\), a balanced \(W\)-separator \(S'\) can be found in time \(2^{O(n)} \cdot \text{poly}(n)\) by enumerating all \(\{S, X, Y\}\) partitions of \(W\) and running the standard network flow algorithm on \(G[V \setminus S]\) to find a separator of \(X\) and \(Y\). This standard approach for finding balanced separator is also used in construction of tree decomposition (see Chap. 11.2 of [29]). Therefore, \(T\) can be constructed in time \(2^{O(n)} \cdot \text{poly}(n)\).

Theorem 4.1 is proved by combining Lemma 4.1 and Theorem 4.2.

5 Counting Algorithms

This section contains three algorithms for Holant problem with regular symmetric constraint functions: a simple exponential-time dynamic programming algorithm; a fixed-parameter tractable (FPT) algorithm which uses the exponential-time algorithm as a subroutine; an FPTAS on apex-minor-free graphs via correlation decay which utilizes the FPT algorithm.

With the construction of separator decomposition, it is not hard to come up with a very natural \(2^{O(nw(G))} \cdot \text{poly}(n)\)-time dynamic programming algorithm for spin systems by enumerating the vertex boundaries and separators of components in the separator decomposition. However, the flexibility of Holant problems causes many new issues to the computation, which require more sophisticated algorithms to deal with.

5.1 A simple \(\exp(O(n))\)-time algorithm. Any Holant problem can be computed in time \(\exp(O(|V|))\) by enumerating all configurations. For Holant problem with regular constraint functions, there is a simple dynamic programming algorithm which runs in time \(\exp(O(|V|))\). This algorithm is used as a subroutine in our main algorithm.

Theorem 5.1. Let \(\Omega = (G(V,E), \{f_v\}_{v \in V})\) be a Holant instance where \(f_v : [q]^{\deg(v)} \to \mathbb{C}\) are symmetric functions. If all \(f_v\) are \(C\)-regular for some constant \(C > 0\), then \(\text{hol}(\Omega)\) can be computed in time \(\exp(qC^{O(n)})\).

We enumerate the vertices in an arbitrary order \(v_1, v_2, \ldots, v_n\). Let \(G_k(V_k, E_k)\) be a subgraph induced by the first \(k\) vertices, i.e. \(V_k = \{v_i \mid 1 \leq i \leq k\}\) and \(E_k = \{(v_i, v_j) \in E \mid 1 \leq i, j \leq k\}\). For a \(v \in V_k\), let \(\deg_k(v)\) denote the degree of \(v\) in \(G_k\). Fix any \(1 \leq k \leq n\). For \(i = 1, 2, \ldots, k\), let \(\phi_{v_i}^{(k)}\) be symmetric functions at vertex \(v_i\) in the form \(\phi_{v_i}^{(k)} : [q]^{\deg_k(v_i)} \to \mathbb{C}\). We define the following quantity:

\[
Z(k, \{\phi_{v_i}^{(k)}\}_{i=1,2,\ldots,k}) = \sum_{\sigma \in [q]^{E_k}} \prod_{i=1}^{k} \phi_{v_i}^{(k)}(\sigma | E_k(v_i)).
\]

In fact, each \(Z(k, \{\phi_{v_i}^{(k)}\}_{i=1,2,\ldots,k})\) defines a new Holant problem on \(G_k\). And the result of the original Holant problem is given by \(\text{hol}(\Omega) = Z(n, \{f_v\}_{i=1,2,\ldots,n})\). In general we have the following recursion:

\[
\begin{align*}
Z(0, \emptyset) &= 1; \\
Z(k, \{\phi_{v_i}^{(k)}\}_{i=1,2,\ldots,k}) &= \sum_{\sigma \in [q]^{E_k(v_k)}} \phi_{v_k}^{(k)}(\sigma) \cdot Z(k-1, \{\phi_{v_i}^{(k-1)}\}_{i=1,2,\ldots,k-1}),
\end{align*}
\]

where \(\phi_{v_i}^{(k-1)} = \{\text{PIN}_{\sigma(v_i, v_k)}^{(k)}(\phi_{v_i}^{(k)})\} \text{ if } v_iv_k \in E_k,\)

otherwise.

This recursion separates the summation into different cases of configurations around \(v_k\) and modifies the functions at the adjacent vertices according to the configuration. The correctness of the recursion can be easily verified by observing that the edge set \(E_k\) is the disjoint union of \(E_{k-1}\) and \(E_k(v_k)\).

We then describe a dynamic programming algorithm which computes the Holant problem \(Z(n, \{f_v\}_{i=1,2,\ldots,n})\) in time \((qC)^{O(n)}\) if all \(f_v\) are \(C\)-regular. The algorithm consists of two phases:

1. Preparation: For every \(v \in V\), construct the set \(\{\text{PIN}_{\sigma}^{(k)}(f_v) \mid \sigma \in [q]^{E}, 0 \leq \ell \leq \deg(v)\}\) which contains all pinning outcomes of \(f_v\). For symmetric \(f_v\) this can be done in time polynomial of \(\deg(v)\).

2. Dynamic programming: It is easy to see that for any \(1 \leq k \leq n\) and any \(1 \leq i \leq k\), function \(\phi_{v_i}^{(k)}\) is an outcome of a sequence of pinning of \(f_{v_i}\). Moreover, it holds that

\[
\phi_{v_i}^{(k)} \in \left\{\text{PIN}_{\sigma}^{(k)}(f_v) \mid \sigma \in [q]^{\deg_k(v_i) - \deg_k(v_i)}\right\},
\]

where the size of the set is bounded by \(C\) since \(f_{v_i}\) is \(C\)-regular. Therefore, \(Z(k, \{\phi_{v_i}^{(k)}\}_{i=1,2,\ldots,k})\) for all \(1 \leq k \leq n\) can be stored in an \(n \times C^n\) table, while each \(\phi_{v_i}^{(k)}\) can be retrieved from \(\{\text{PIN}_{\sigma}^{(k)}(f_v) \mid \sigma \in [q]^{\deg_k(v_i) - \deg_k(v_i)}\}\) by an index ranging over \([C]\). It takes at most \(q^{O(n)}\) time to fill each entry of the table. The total time complexity is \((qC)^{O(n)}\).
5.2 A fixed-parameter tractable algorithm.

**Theorem 5.2.** Let $\Omega = (G(V, E), \{f_v\}_{v \in V})$ be a Holant instance where $f_v : [q]^{\deg(v)} \to \mathbb{C}$ are symmetric functions. If all $f_v$ are C-regular for some constant $C > 0$, then $\text{hol}(\Omega)$ can be computed in time $2^{O(tw(G))} \cdot \text{poly}(n)$ where $tw(G)$ represents the treewidth of $G$.

The $2^{O(tw(G))}$ growth in treewidth is critical to our approximation algorithm on planar graphs introduced later, although any faster growth in treewidth is still fixed-parameter tractable.

**The setup.** Suppose that $G(U \cup \partial U, F)$ be the subgraph that $F = \{uv \in E \mid u, v \in U$ or $u \in U, v \in \partial U\}$, i.e., $F$ includes all edges within $U$ and all edges crossing between $U$ and $\partial U$ (but not those edges with both endpoints in $\partial U$). For each $v \in \partial U$, let $\phi_v : [q]^{\deg(H)} \to \{0, 1\}$ be a boolean symmetric function. We define the following quantity:

\[
Z(U, \{\phi_v\}_{v \in \partial U}) = \sum_{\sigma \in [q]^F} \prod_{v \in U \cup \partial U} g_v(\sigma | F(v)),
\]

where $g_v = f_v$ if $v \in U$, $\phi_v$ if $v \in \partial U$.

In fact $Z(U, \{\phi_v\}_{v \in \partial U})$ defines a Holant problem on graph $H(U \cup \partial U, F)$ with function $f_v$ at each $v \in U$ and boolean constraint $\phi_v$ at each boundary vertex $v \in \partial U$. And the original Holant problem can be written as $\text{hol}(\Omega) = Z(V, \emptyset)$.

**The recursion.** Suppose that $U$ can be partitioned into $S, U_1, U_2$ such that $S$ is a $(U_1, U_2)$-separator of $U$ in $G[U]$, where $G[U]$ is the subgraph of $G$ induced by $U$ (note that $S$ is not necessarily a separator of $H$). It is obvious that $\partial U_1 \subseteq S \cup \partial U$ and $\partial U_2 \subseteq S \cup \partial U$, where $\partial U_1$ and $\partial U_2$ are respectively the vertex boundaries of $U_1$ and $U_2$ in $G$. For each $v \in S \cup \partial U$, let $d_0(v), d_1(v), d_2(v)$ denote the number neighbors of $v$ in $S \cup \partial U, U_1, U_2$ respectively that is,

\[
d_0(v) = |\{u \in S \cup \partial U \mid uv \in F\}|,
\]

\[
d_1(v) = |\{u \in U_1 \mid uv \in F\}|,
\]

\[
d_2(v) = |\{u \in U_2 \mid uv \in F\}|.
\]

It holds that $d_0(v) + d_1(v) + d_2(v) = \deg_H(v)$.

For any $v \in S \cup \partial U$ and $i = 0, 1, 2$, each tuple $\sigma \in [q]^{d_i(v)}$ can be mapped to a boolean function $\text{Peer}_{\sigma}(g_v)$ which indicates all tuples that have the same effect of pinning on $g_v$ as $\sigma$, where $g_v$ is still defined as that $g_v = f_v$ for $v \in S$ and $g_v = \phi_v$ for $v \in \partial U$. We call $\text{Peer}_{\sigma}(g_v)$ the peer image of $\sigma$ at $v$. For each $v \in S \cup \partial U$ and $i = 0, 1, 2$, let $P_i^v = \{\text{Peer}_{\sigma}(g_v) \mid \sigma \in [q]^{d_i(v)}\}$ be the range of peer images over all $\sigma \in [q]^{d_i(v)}$. Let $\phi$ be a sequence indexed by $\phi_v$ for $v \in S \cup \partial U$ and $i = 0, 1, 2$, such that $\phi_v \in P_i^v$, i.e., $\phi_v$ is a peer image of some $\sigma \in [q]^{d_i(v)}$.

We then have the following recursion for the quantity $Z(U, \{\phi_v\}_{v \in \partial U})$ defined in (5.1):

\[
Z(U, \{\phi_v\}_{v \in \partial U}) = \sum_{\phi_v \in P_i^v, \forall v \in S \cup \partial U} Z_2(\phi) \prod_{v \in S \cup \partial U} \tilde{g}_v(\phi_v, \phi_v, \phi_v),
\]

where

\[
Z_0(\phi) = \text{hol}(H[S \cup \partial U], \{\phi_v\}_{v \in S \cup \partial U}),
\]

\[
Z_1(\phi) = Z(U_1, \{\phi_v\}_{v \in U_1}),
\]

\[
Z_2(\phi) = Z(U_2, \{\phi_v\}_{v \in U_2}),
\]

and

\[
\tilde{g}_v(\phi_v, \phi_v, \phi_v) = g_v(\sigma_0\sigma_1\sigma_2)
\]

for arbitrary $(\sigma_0, \sigma_1, \sigma_2)$ with $\text{Peer}_{\sigma_i}(g_v) = \phi_v^i$ for $i = 0, 1, 2$.

Due to Lemma 3.1, the value of $g_v(\sigma_0\sigma_1\sigma_2)$ is uniquely determined by the peer images $\text{Peer}_{\sigma_i}(g_v), i = 0, 1, 2$, thus $\tilde{g}_v$ is well defined. Peer images $\phi_v$ are boolean functions, thus $Z_0, Z_1, Z_2$ are also well defined.

As an example, consider counting matchings, which is a Holant problem of regular constraint functions. The peer images $\phi_v$ actually correspond to that vertex $v$ is matched or unmatched by the corresponding subset of incident edges of $v$ (we ignore the overmatched case because it nullifies the configuration.). The Holant problem $Z_0(\phi) = \text{hol}(H[S \cup \partial U], \{\phi_v\}_{v \in S \cup \partial U})$ counts the number of perfect matchings of those vertices that claim to be matched in $H[S \cup \partial U]$.

We then prove that the recursion (5.2) holds for the quantity $Z(U, \{\phi_v\}_{v \in \partial U})$ defined by (5.1).

**Proof.** Since $S$ is a $(U_1, U_2)$-separator of $U$, the set of all edges in the original subgraph $H(U \cup \partial U, F)$ can be partitioned into five disjoint sets:

\[
F_0 = \{uv \in F \mid u \in S, v \in S \cup \partial U\},
\]

for $i = 1, 2, F_i = \{uv \in F \mid u, v \in U_i\}$, for $i = 1, 2, F_i = \{uv \in F \mid u \in U_i, v \in S \cup \partial U\}$.
That is, $F_0$ is the set of internal edges of $S \cup \partial U$, $E_i$ is the set of internal edges of $U_i$, and $F_i$ is the set of boundary edges of $U_i$, $i = 1, 2$.

Each vertex $v \in S \cup \partial U$ has precisely $d_i(v)$ adjacent edges in $F_i$ for $i = 0, 1, 2$. And for $i = 1, 2$, $\partial U_i$ is precisely the set of vertices in $S \cup \partial U$ with positive $d_i(v)$. We can enumerate all configurations $\sigma \in [q]^{F_0 \cup F_1 \cup F_2}$ by enumerating legal local configurations $\sigma_i^v \in [q]^{F_i(v)}$ for each individual vertex $v \in S \cup \partial U$ and each $i = 0, 1, 2$, where being legal means that there exists a $\sigma \in [q]^{F_0 \cup F_1 \cup F_2}$ such that $\sigma |_{F_i(v)} = \sigma_i^v$ for all $v \in S \cup \partial U$ and $i = 0, 1, 2$.

For a tuple $\sigma \in [q]^k$, we define the indicator function $1_\sigma : [q]^k \to \{0, 1\}$ as that $1_\sigma(\tau) = 1$ if and only if $\tau = \sigma$. Then (5.1) can be trivially rewritten as follows:

$$Z(U, \{\phi_v\}_{v \in \partial U}) = \sum_{\sigma \in [q]^{F_0 \cup F_1 \cup F_2}} Z(U_1, \{1_{\sigma |_{F_1(v)}}\}_{v \in \partial U_1}) \cdot Z(U_2, \{1_{\sigma |_{F_2(v)}}\}_{v \in \partial U_2}) \cdot \prod_{v \in S \cup \partial U} g_v(\sigma |_{F_0(v) \cup F_1(v) \cup F_2(v)}).$$

In fact, $F_1$ can be partitioned into disjoint $F_1(v)$ for $v \in \partial U_1$ and $F_2$ can be partitioned into disjoint $F_2(v)$ for $v \in \partial U_2$. Thus for $i = 1, 2$ all local configurations $\{\sigma_i^v \in [q]^{d_i(v)}\}_{v \in S \cup \partial U}$ are legal. A collection $\{\sigma_i^v \in [q]^{d_i(v)}\}_{v \in S \cup \partial U}$ of local configurations of edges in $F_0$ of individual vertices is legal if and only if the Holant problem

$$\text{hol}(H[S \cup \partial U], \{1_{\sigma_i^v}\}_{v \in S \cup \partial U})$$

has value 1 (it has only two possible values 0 or 1 as every indicator function has value 1 on exactly one input). Thus we have

$$Z(U, \{\phi_v\}_{v \in \partial U}) = \sum_{\sigma_i^v \in [q]^{d_i(v)}} \text{hol}(H[S \cup \partial U], \{1_{\sigma_i^v}\}_{v \in S \cup \partial U}) \cdot Z(U_1, \{1_{\sigma_i^v}\}_{v \in \partial U_1}) \cdot Z(U_2, \{1_{\sigma_i^v}\}_{v \in \partial U_2}) \cdot \prod_{v \in S \cup \partial U} g_v(\sigma_i^v |_{F_0(v) \cup F_1(v) \cup F_2(v)}).$$

For $v \in S \cup \partial U$ and $i = 0, 1, 2$, fix $\phi_i^v \in \mathcal{P}_i$, i.e. $\phi_i^v = \text{PEER}_\sigma(g_v)$ for some $\sigma \in [q]^{d_i(v)}$. We can group configurations $\{\sigma_i^v\}_{v \in S \cup \partial U, i = 0, 1, 2}$ into equivalence classes $\{\sigma_i^v \in [q]^{d_i(v)} \mid \text{PEER}_\sigma(g_v) = \phi_i^v\}$ according to their peer images. Due to Lemma 3.1, configurations from the same class yields the same value of $g_v(\sigma_i^v |_{F_0(v) \cup F_1(v) \cup F_2(v)})$.

Therefore,

$$Z(U, \{\phi_v\}_{v \in \partial U}) = \sum_{\phi_i^v \in \mathcal{P}_i, \phi_j^v \in \mathcal{P}_j} Z_0(\phi) \cdot Z_1(\phi) \cdot Z_2(\phi) \cdot \prod_{v \in S \cup \partial U} g_v(\phi_i^v, \phi_j^v, \phi_k^v),$$

where

$$Z_0(\phi) = \sum_{\phi_i^v \in [q]^{d_i(v)}, \text{PEER}_\sigma(g_v) = \phi_i^v} \text{hol}(H[S \cup \partial U], \{1_{\phi_i^v}\}_{v \in S \cup \partial U}),$$

$$Z_1(\phi) = \sum_{\phi_i^v \in [q]^{d_i(v)}, \text{PEER}_\sigma(g_v) = \phi_i^v} Z(U_1, \{1_{\phi_i^v}\}_{v \in \partial U_1}),$$

$$Z_2(\phi) = \sum_{\phi_i^v \in [q]^{d_i(v)}, \text{PEER}_\sigma(g_v) = \phi_i^v} Z(U_2, \{1_{\phi_i^v}\}_{v \in \partial U_2}).$$

Note that any peer image $\phi_i^v \in \mathcal{P}_i$ is a boolean function which indicates all such $\sigma \in [q]^{d_i(v)}$ that have the same peer image PEER$_\sigma(g_v) = \phi_i^v$, thus it is straightforward to verify the following identities:

$$Z_0(\phi) = \text{hol}(H[S \cup \partial U], \{\phi_i^v\}_{v \in S \cup \partial U}),$$

$$Z_1(\phi) = \text{Z}(U_1, \{\phi_i^v\}_{v \in \partial U_1}),$$

$$Z_2(\phi) = \text{Z}(U_2, \{\phi_i^v\}_{v \in \partial U_2}).$$

Substituting these identities back in (5.3), we have the recursion (5.2).

**The algorithm.** We then describe an algorithm which computes $\text{hol}(G(V, E), \{f_v\}_{v \in V})$ in time $2^{O(tw(G))} \cdot \text{poly}(|V|)$ if all $f_v$ are C-regular for some constant $C > 0$.

1. Construct a separator decomposition of width at most $O(tw(G))$ in time $2^{O(tw(G))} \cdot \text{poly}(n)$ (Theorem 4.1).
2. Construct the peering closures. For every $v \in V$ and each $0 \leq k \leq \text{deg}(v)$, construct set...
Dynamic programming: Let $T$ be the separator decomposition constructed in the first step. Then each node $i \in T$ associated with a vertex set $V_i$ and a separator $S_i$ such that $|S_i| = O(\text{tw}(G))$ and $|\partial V_i| = O(\text{tw}(G))$, and if $j$ and $k$ are the two children of $i$ in $T$, $S_i$ is a $(V_j, V_k)$-separator in $G[V_i]$. Apply the recursion (5.2) in this tree structure as follows: For each leaf $\ell \in T$, $V_\ell = \emptyset$, and $Z(V_\ell, \emptyset) = 1$; and for each non-leaf node $i \in T$ with children $j$ and $k$ in $T$, $Z(V_i, \{\phi_v\}_{v \in \partial V_i})$ is computed according to the recursion (5.2) by setting $U = V_i$, $U_1 = V_j$, and $U_2 = V_k$; in particular for the root $r$ of $T$, $V_r = V$ and $Z(V_r, \emptyset) = \text{hol}(G(V, E), \{f_v\}_{v \in V})$.

There are $O(|V|)$ nodes in a separator decomposition. For all $Z(V_i, \{\phi_v\}_{v \in \partial V_i})$, every $\phi_v$ is a boolean function in $\text{PEER}^*(f_v)$. Due to Lemma 3.4, since $f_v$ is $C$-regular, once $V_i$ is fixed there are at most $2^C$ possible $\phi_v$ for each $v \in \partial V_i$, where $|\partial V_i| = \text{tw}(G)$. Therefore all $Z(V_i, \{\phi_v\}_{v \in \partial V_i})$ can be stored in an $O(n) \times 2^{O(C \cdot \text{tw}(G))}$ table.

Each entry of the dynamic programming table is filled according to the recursion (5.2), which involves three nontrivial tasks:

(a) (computing $Z_0$): Due to Lemma 3.4, any $\phi_v \in \text{PEER}^*(f_v)$ is still $C$-regular since $f_v$ is $C$-regular, thus $Z_0 = \text{hol}(H_i[S_i \cup \partial V_i], \{\phi_v\}_{v \in S_i \cup \partial V_i})$ is a Holant problem with $C$-regular constraint functions which can be computed in time $(qC)^{|S_i \cup \partial V_i|} = (qC)^{O(\text{tw}(G))}$ by Theorem 5.1.

(b) (evaluating $\tilde{g}_v$): Each $\tilde{g}_v(\phi_v^0, \phi_v^1, \phi_v^2)$ can be easily evaluated by evaluating $g_v(\sigma_0 \sigma_1 \sigma_2)$ for arbitrary $\sigma_0 \in \phi_v^0, \sigma_1 \in \phi_v^1, \sigma_2 \in \phi_v^2$.

(c) (computing the sum): For every $v \in S_i \cup \partial V_i$, enumerate all at most $2^C$ possible boolean functions of appropriate arity $\phi_v \in \text{PEER}^*(f_v)$. The total time is bounded by $2^{O(C \cdot \text{tw}(G))}$ because $|S_i \cup \partial V_i| = O(\text{tw}(G))$.

The time cost for filling one entry of the dynamic programming table is bounded by $2^{O(\text{tw}(G))}$ for constant $C$ and $q$.

The total time cost for the above algorithm is bounded by $2^{O(\text{tw}(G))} \cdot \text{poly}(n)$ for constant $C$ and $q$.

5.3 An FPTAS from correlation decay.

**Theorem 5.3.** Assume the tractable search for the Holant problem $\text{Holant}(G, F)$ where $G$ is an apex-minor-free graph family and $F$ is a regular family of nonnegative symmetric functions. The strong spatial mixing implies the existence of FPTAS for $\text{Holant}(G, F)$.

Let $\Omega = (G(V, E), \{f_v\}_{v \in V})$ be a Holant instance, where $G$ is an apex-minor-free graph and all $f_v : [q]^{\text{deg}(v)} \to \mathbb{R}^+$ are $C$-regular symmetric functions for some constant $C > 0$. Let $\mu$ be the Gibbs measure defined by the Holant instance $\Omega$.

Assume the tractable search and strong spatial mixing for $\text{Holant}(G, F)$. We have the following lemma for approximation of marginal probabilities.

**Lemma 5.1.** Let $e \in E$. Let $\Lambda \subseteq E$ and $\tau_\Lambda \in [q]^\Lambda$ be a feasible configuration. The marginal probability $\mu^{\tau_\Lambda}(i)$ for any $i \in [q]$ can be approximated within any additive error $\epsilon$ in time $\text{poly}(n, \frac{1}{\epsilon})$.

**Proof.** Let $N_r(e) = \{e' \in E \mid \text{dist}(e, e') \leq r\}$ be the $r$-neighborhood of edge $e$ in $G$. Let $B_r(e) = \{uv \in E \setminus N_r(e) \mid \exists wv \in N_r(e)\}$ be the edge boundary of the $r$-neighborhood.

Denote $\Delta = B_r(e) \setminus \Lambda$. When the strong spatial mixing holds, by Definition 2.3, for any $\sigma = \{\sigma_i\}_{i \in \Lambda} \Delta$ such that both $\tau_\Lambda$ and $\tau_\Delta$ are feasible, it holds that $\|\mu^{\tau_\Lambda} - \mu^{\tau_\Delta}\|_\text{TV} \leq \text{poly}(n) \cdot \exp(O(-r))$. Therefore for any $\sigma = \{\sigma_i\}_{i \in \Lambda}$ such that $\tau_\Lambda = \tau_\Delta$ is feasible, we have

$$\|\mu^{\tau_\Lambda} - \mu^{\tau_\Delta}\|_\text{TV} \leq \text{poly}(n) \cdot \exp(O(r)),$$

because $\mu^{\tau_\Lambda}$ is a linear combination of all such $\mu^{\tau_\Delta}$. Note that the joint configuration $\tau_\Lambda, \tau_\Delta$ fixes the boundary $B_r(e)$. Thus for each $i \in [q]$, the marginal probability $\mu_e^{\tau_\Lambda}(i)$ can be computed precisely from the $r$-neighborhood as follows:

Let $W$ be the set of incident vertices of $N_r(e)$ and $F = N_r(e) \setminus \Lambda$. Let $H(W, F)$ be the subgraph formed by removing edges fixed by $\sigma_i$ from the $r$-neighborhood. For $i \in [q]$, let $e \mapsto i$ denote the configuration on $\{e\}$ that simply assigns value $i$ to edge $e$. We have that

$$\mu_e^{\tau_\Lambda}(i) = \frac{\text{hol}(H'(W, F \setminus \{e\}), \{f_v^{\tau_\Lambda, \sigma, \tau_\Delta}(i)\}_{v \in W})}{\text{hol}(H(W, F), \{f_v^{\tau_\Lambda, \sigma, \tau_\Delta}(i)\}_{v \in W})},$$

where $f_v^{\tau_\Lambda, \sigma, \tau_\Delta}(i) = \text{PIN}_{\tau_\Lambda}(f_v)$ and $\tau_\Delta = \tau |_{F\setminus\{e\}}$.

The correctness of the equation and the well-definedness of the new Holant problems are easy to verify.

Since the original graph $G$ is apex-minor-free, due to Theorem 2.1 we have $\text{tw}(H) = O(r)$. Since all original
for any feasible $\tau_\Lambda \in \{q\}^\Lambda$ it is possible to efficiently choose an arbitrary feasible $\sigma \in \{q\}^E$ agreeing with $\tau_\Lambda$. Thus a feasible $\sigma_\Delta \in \{q\}^\Delta$ can be efficiently constructed by restricting the aforementioned $\sigma$ on $\Delta$.

Due to (5.4), the original marginal probability $\mu_{\tau_\Delta}^\epsilon(i)$ for any $i \in [q]$ can be approximated within an additive error $\epsilon$ in time $poly(n, \frac{1}{\epsilon})$ by choosing appropriate $r = O((log n + log \frac{1}{\epsilon}))$. We further remark that with such choice of $r$, for some $G$ the $r$-neighborhood $N_r(e)$ might already contain the entire $G$, in which case the set $\Delta = \emptyset$ and the value of marginal probability $\mu_{\tau_\Delta}^\epsilon(i)$ is computed exactly by the algorithm.

With the above lemma, we can apply the standard self-reduction procedure to obtain the FPTAS for Holant$(G, \mathcal{F})$.

Let $\tau \in \{q\}^E$ be a feasible configuration, i.e. the Gibbs measure $\mu(\tau) > 0$. Enumerate edges in $E$ as $e_1, e_2, \ldots, e_m$. For each $0 = k \leq m$, let $E_k = \{e_1, \ldots, e_k\}$, $\tau_k \in \{q\}^{E_k}$ be consistent with $\tau$ on $E_k$, and $p_k = \mu_{\tau_k}^{-1}(\tau(e_k))$. The following identity hold for $\mu(\tau)$:

$$
\mu(\tau) = \prod_{k=1}^{m} \Pr_{\epsilon \in \{q\}^E}[\sigma(e_k) = \tau(e_k) \mid \sigma(e_i) = \tau(e_i), i < k] = \prod_{k=1}^{m} p_k.
$$

On the other hand, $\mu(\tau) = \frac{\prod_{v \in V} f_v(\tau_{|R(v)})}{\text{hol}(\Omega)}$. Thus $\text{hol}(\Omega) = \frac{\prod_{v \in V} f_v(\tau_{|R(v)})}{\prod_{k=1}^{m} p_k}$. If for each $k$: (1) $p_k$ can be approximated in an additive error $\epsilon$; and (2) $p_k > 0$ is a constant, then the product $\prod_{k=1}^{m} p_k$ can be approximated within a multiplicative factor $(1 \pm O(\epsilon))$.

While (1) is guaranteed by Lemma 5.1, (2) can be achieved by trying $\mu_{\tau_k}^{-1}(i)$ for all $i \in [q]$ and choosing $\tau(e_k)$ to be the $i$ with the largest returned value. This gives us an FPTAS for the Holant problem.

We then can directly apply any known strong spatial mixing result to get the FPTAS. For example, combining with the result of [35], we have the following corollary.

**Corollary 5.1.** There exists an FPTAS for counting $q$-coloring on apex-minor-free triangle-free graphs of maximum degree at most $\Delta$ if $q > \alpha \Delta - \gamma$ where $\alpha \approx 1.76322$ is the solution to $\alpha^3 = e$ and $\gamma = \frac{4\alpha^3 - 6\alpha^2 - 3\alpha + 1}{2(\alpha - 1)} \approx 0.47031$.

Note that although the original result of [35] is proved for single-site strong spatial mixing where the boundaries differ on only one vertex, it implies the definition of strong spatial mixing on any finite graphs.

### 6 Correlation Decay

In this section we apply the recursive coupling technique [35] to Holant problems, and prove strong spatial mixing for subgraphs world [36] and ferromagnetic Potts model. The algorithmic implications of these correlation decay results are presented in the end of this section.

#### 6.1 Recursive coupling on Holant Problems

Consider a Holant problem $\text{Holant}(G, \mathcal{F})$ and an instance $\Omega = (G(V, E), \{f_v\}_{v \in V})$. Let $R \subseteq E$, called region. Define $\delta R = \{w \in E \mid w \not\in R, 3w \not\in R\}$ the edge boundary of $R$. Define $V_R = \{v \in V \mid 3w \not\in R\}$.

A boundary configuration of $R$, is a $\sigma \in \{q\}^{\delta R}$. For every boundary configuration $\sigma \in \{q\}^{\delta R}$ and a configuration $\eta \in \{q\}^R$ of the region $R$, define the regional weight as

$$
w_R^\sigma(\eta) = \prod_{v \in V_R} f_v(\eta \mid R(v)) \sigma(\mid \delta R(v))
$$

where $\eta \mid R(v)$ is the restriction of $\eta$ on the edges in $R$ incident to $v$, $\sigma(\mid \delta R(v))$ is the restriction of $\sigma$ on edges in $\delta R$ incident to $v$, and $f_v(\sigma \mid R(v)) \eta(\mid \delta R(v))$ evaluates $f_v$ on the concatenation of them.

We say that a boundary configuration $\sigma \in \{q\}^{\delta R}$ is $R$-feasible if there exists an $\eta \in \{q\}^R$ such that $w_R^\sigma(\eta) > 0$. For $R$-feasible boundary configuration $\sigma \in \{q\}^{\delta R}$, a regional Gibbs measure $\mu_R^\sigma$ over $\{q\}^R$ can be defined as $\mu_R^\sigma(\eta) = \frac{w_R^\sigma(\eta)}{\sum_{\eta' \in \{q\}^R} w_R^{\sigma}(\eta')}$ for each $\eta \in \{q\}^R$.

For $R' \subseteq R$, let $\mu_{R',R}^\sigma$ denote the marginal distribution of $\mu_R^\sigma$ on $R'$, and we write that $\mu_{R',R}^\sigma = \mu_{R',e}^\sigma$.

**Definition 6.1.** Let $R \subseteq E$, $R' \subseteq R$, and $\sigma, \tau \in \{q\}^{\delta R}$ be two $R$-feasible boundary configurations. Let $\Psi(\mathcal{R}, \sigma, \tau)$ be a coupling of $\mu_R^\sigma, \mu_R^\tau$. Define the discrepancy of $\Psi(\mathcal{R}, \sigma, \tau)$ on region $R' \subseteq R$ as

$$
\text{Disc}_{\Psi(\mathcal{R}, \sigma, \tau)}(R') = \Pr_{(\eta, \eta') \sim \Psi(\mathcal{R}, \sigma, \tau)}[\eta \mid R' \neq \eta' \mid R']
$$

We write that $\text{Disc}_{\Psi(\mathcal{R}, \sigma, \tau)}(e) = \text{Disc}_{\Psi(\mathcal{R}, \sigma, \tau)}(\{e\})$.

**Definition 6.2.** For any two $R$-feasible boundary configurations $\sigma, \tau \in \{q\}^{\delta R}$ differing on $\Delta \subseteq \delta R$, a sequence of $R$-feasible boundary configurations $\sigma_1, \sigma_2, \ldots, \sigma_t$ is
called a feasible path from $\sigma$ to $\tau$ if $\sigma = \sigma_1, \tau = \sigma_T$ and $\sigma_i, \sigma_{i+1}$ differ only at one edge $e \in \Delta$ for each $1 \leq i < T$. Let $T(\sigma, \tau)$ be the minimum such $T$, or be $\infty$ if no such path exists.

**Lemma 6.1.** Let $\Lambda \subseteq E$ and $\sigma, \tau \in [q]^{\Lambda}$ be two feasible configurations differing on $\Delta \subseteq \Lambda$. Let $R = E \setminus \Lambda$ and $e \in R$. There exist two $R$-feasible boundary configurations $\sigma', \tau' \in [q]^{\delta R}$ differing only on edges in $\Delta$ such that

$$\|\mu_{R,e} - \mu_{\tau,R,e}\|_{TV} \leq T(\sigma', \tau') \cdot \max_{\sigma, \tau \in [q]^R \text{ differ on } e' \in \Delta} \|\mu_{\sigma,e} - \mu_{\tau,e}\|_{TV},$$

for arbitrary coupling $\Psi(R, \sigma, \tau)$ of $\mu_{R}^{\tau}, \mu_{\tau}^{\sigma}$.

**Proof.** Let $\sigma', \tau' \in [q]^{\delta R}$ consistent with $\sigma, \tau$ on $\delta R$ respectively. It is easy to check that $\sigma', \tau'$ satisfy the equation. Let $\sigma' = \sigma_1, \sigma_2, \ldots, \sigma_T = \tau'$. Then, $\sigma' < T(\sigma', \tau')$ and $\|\mu_{R,e} - \mu_{\tau,R,e}\|_{TV}$ can be bounded by applying path coupling to $\mu_{R,e}^{\tau}, \mu_{\tau,R,e}^{\sigma}$ for $1 \leq i < T(\sigma', \tau')$. The last inequality is due to the coupling lemma.

The above lemma reduces the strong spatial mixing to the discrepancy witnessed by a coupling of $\mu_{\tau,R}^{\sigma}, \mu_{\sigma,R}^{\tau}$ with $\sigma, \tau$ disagreeing at one edge. We then show a way to recursively construct the coupling. This method is proposed by Goldberg et al. in [35] on colorings.

### The recursive coupling

Let $R \subseteq E$ and $e_0 \in R$. Let $\sigma, \tau \in [q]^{\delta R}$ be any two $R$-feasible boundary configurations that differ at only one edge $e \in \delta R$. Let $R(e)$ be the set of edges in $R$ incident to $e$. Let $\Psi_{R(e)}(R, \sigma, \tau)$ be a coupling of marginal distributions $\mu_{R,R(e)}^{\sigma}, \mu_{R,R(e)}^{\tau}$. A coupling $\Psi(R, \sigma, \tau)$ of regional Gibbs measures $\mu_{R}^{\sigma}, \mu_{R}^{\tau}$ can be recursively constructed by the local coupling rule $\Psi_{R(e)}(R, \sigma, \tau)$. Let $(\eta, \eta') \in [q]^R \times [q]^R$ denote the pair sampled from $\Psi_{R(e)}(R, \sigma, \tau)$.

1. **(Base case)** If $e_0 \in R(e)$, sample $(\eta \mid R(e), \eta' \mid R(e))$ according to $\Psi_{R(e)}(R, \sigma, \tau)$ and arbitrarily sample the rest of $(\eta, \eta')$ conditioning on $(\eta \mid R(e), \eta' \mid R(e))$ as long as $(\eta, \eta')$ is a faithful coupling of $\mu_{R}^{\sigma}, \mu_{R}^{\tau}$. If $|R(e)| = 0$, which case $e$ and $e_0$ are disconnected in $G[V_0]$, sample $(\eta, \eta')$ such that $(\eta(e_0), \eta'(e_0))$ is perfectly coupled.

2. **(General case)** If $|R(e)| > 0$ and $e_0 \notin R(e)$. Sample $(\eta \mid R(e), \eta' \mid R(e))$ according to $\Psi_{R(e)}(R, \sigma, \tau)$. Let $x, y \in [q]^{R(e)}$ be two configurations that $x = \eta \mid R(e), y = \eta' \mid R(e)$. Construct new region and boundaries as: $R' = R \setminus R(e); \sigma' \in [q]^{\delta R'}$ agrees with $\sigma$ on common edges and $\sigma' \mid R(e) = x$; and $\tau' \in [q]^{\delta R'}$ agrees with $\tau$ on common edges and $\tau' \mid R(e) = y$. The rest of $(\eta, \eta')$ is sampled from a coupling $\Psi(x, y)$ of $\mu_{R'}^{\sigma'}, \mu_{R'}^{\tau'}$. If $x = y$, then $x = \tau'$ and $\Psi(x, y)$ is a perfect coupling. If $x \neq y$, let $\sigma_1(x, y), \ldots, \sigma_2(x, y)$ be a feasible path from $\sigma'$ to $\tau'$ of length $t = T(\sigma', \tau')$. Let $\Psi(x, y)$ be the composition of coupling $\Psi \left( R', \sigma_1(x, y), \sigma_2(x, y) \right)$, i.e. $i = 1, 2, \ldots, t - 1$, in the same manner as path coupling, where each $\Psi \left( R', \sigma_1(x, y), \sigma_2(x, y) \right)$ can be recursively defined as $\sigma_1(x, y)$ and $\sigma_2(x, y)$ differ at only one edge. It is easy to verify that $\Psi(x, y)$ is a coupling of $\mu_{R'}^{\sigma'}, \mu_{R'}^{\tau'}$. This complete the construction of $\Psi(R, \sigma, \tau)$.

The following lemma is similar to the one proved in [35] for the recursive coupling constructed on spin systems.

**Lemma 6.2.** For the coupling $\Psi(R, \sigma, \tau)$ constructed as above, we have

$$\text{Disc}_{\Psi_{R(\sigma, \tau)}}(e_0) \leq \sum_{x, y \in [q]^{R(e)}} \Pr_{(\eta, \eta') \sim \Psi(x, y)} \left( \eta \mid R(e) = x \wedge \eta' \mid R(e) = y \right) T(\sigma', \tau') - 1 \sum_{i = 1} \text{Disc}_{\Psi_{R(e)(i), \sigma_1, \sigma_2}}(e_0) \cdot \max_{\sigma_1, \sigma_2 \in [q]^{\delta R(e)}} \text{Disc}_{\Psi_{R(\sigma, \tau)}}(e_0) \cdot \max_{\sigma_1, \sigma_2 \in [q]^{\delta R(e)}} \text{Disc}_{\Psi_{R(\sigma, \tau)}}(e_0) \cdot \max_{\sigma_1, \sigma_2 \in [q]^{\delta R(e)}} \text{Disc}_{\Psi_{R(\sigma, \tau)}}(e_0) \cdot \max_{\sigma_1, \sigma_2 \in [q]^{\delta R(e)}} \text{Disc}_{\Psi_{R(\sigma, \tau)}}(e_0)$$

**Proof.** The lemma follows directly from our construction of $\Psi(R, \sigma, \tau)$.

A standard choice of $\Psi_{R(e)}(R, \sigma, \tau)$ is the one that the probability $\Pr(\eta \mid R(e) \sim \Psi_{R(e)}(R, \sigma, \tau)) = \eta \mid R(e) = \eta' \mid R(e)$ is maximized, i.e. $\text{Disc}_{\Psi_{R(e)}(R, \sigma, \tau)}(R(e))$ is minimized.

### 6.2 The subgraphs world of Ising model

The subgraphs world model used in [36] for developing FPRAS for the ferromagnetic Ising model, is a counting problem computationally equivalent to the Ising model under holographic transformation.

**Definition 6.3.** (Subgraphs world [36]) The subgraphs world with parameters $(\lambda, \mu)$ defined as follows.
Let $G = (V, E)$ be an undirected graph. The subgraphs world partition function is defined as:

$$Z_{\text{sub}}(G) = \sum_{X \subseteq E} \mu^{\text{odd}(X)} \chi^{|X|},$$

where $\text{odd}(X)$ denotes the set of vertices with odd degree in the subgraph $(V, X)$.

The subgraphs world with parameter $(\lambda, \mu)$ can be interpreted as a Holant problem on incident graph $I_G$ as follows. The incident graph $I_G$ has left vertex set $V$ and right vertex set $E$, and for each $v \in V$ and $e \in E$, $(v, e)$ is an edge in $I_G$ if $e$ is incident to $v$ in $G$. The function on each left vertex $v$ is $[1, \mu, 1, \mu, \ldots]$ and the function on each right vertex $e$ is $[1, 0, \lambda]$. Let $\Omega$ be the Holant instance defined above. It is easy to verify that all functions in $\Omega$ are 3-regular and $Z_{\text{sub}}(G) = \hol(\Omega)$.

For convenience of analysis, we consider the following equivalent Holant problem which is defined on the original graph $G$ instead of the incident graph. Let $\Omega' = \{(G, E), \{f_v\}_{v \in V}\}$ be a Holant instance where each $f_v = \{f_0, f_1, \ldots, f_{\deg(v)}\}$ has that $f_k = \mu \lambda^{k/2}$ if $k$ is odd and $f_k = \lambda^{k/2}$ if $k$ is even. Note that although this parameterization of subgraphs world is no longer defined by regular constraints, we have that $\hol(\Omega) = \hol(\Omega')$ and also this parameterizations has exact the same Gibbs measure as the original one. Thus the SSM of this parameterization implies the SSM of the original subgraphs world problem defined by regular constraints.

**Theorem 6.1.** Let Holant$(G, F)$ be defined by the subgraphs world of parameter $(\mu, \lambda)$ with $0 < \mu, \lambda < 1$ on graphs with degree bound $\Delta$. If $\Delta < \frac{(1+\lambda\mu^2)^2}{1-\mu^2}$ then Holant$(G, F)$ has strong spatial mixing.

**Proof.** Let $R \subseteq E$ be a region and $\sigma, \tau \in \{0, 1\}^{4R}$ be two $R$-feasible boundary configurations differing at $e \in 6R$ satisfying $\sigma(e) = 0$ and $\tau(e) = 1$. The regional weights $w_R^R, w_R^H$ and regional Gibbs measures $\mu_R^R, \mu_R^H$ can be defined accordingly.

Let $\Psi_{R(o)}(R, \sigma, \tau)$ be the coupling of the joint distribution $(\mu_{R(o)} R, \tau_{R(o)} R, \rho_{R(o)} R)$ such that the discrepancy $\text{Disc}_{\Psi_{R(o)}}(R, \sigma, \tau)(R(e))$ is minimized, i.e., the probability $\Pr((\eta, \eta')) \sim \Psi_{R(o)}(R, \sigma, \tau)[\eta][R(e)] = \Pr(\eta)[R(e)]$ is maximized. We first give an upper bound on $\text{Disc}_{\Psi_{R(o)}}(R, \sigma, \tau)(R(e))$.

Let $\eta \in \{0, 1\}^R$ be a configuration on $R$, we use $n_\eta$ to denote the number of edges in $R(e)$ that are assigned to 1 by $\eta$. Let

$$w_e = \sum_{\eta \in \{0, 1\}^R} w_{R(o)} \eta^n \eta \text{ is even}, \quad w_o = \sum_{\eta \in \{0, 1\}^R} w_{R(o)} \eta^n \eta \text{ is odd},$$

For every $\eta \in \{0, 1\}^R$, we denote that $w_{\eta} = w_{R(o)}^H(\eta)$ thus we have $\mu_{R(o)}(\eta) = w_{\eta} w_o - w_{\eta} w_o$ and

$$\mu_{R(o)}(\eta) = \begin{cases} w_{\eta} w_o - w_{\eta} w_o & \text{if } n_\eta \text{ is even,} \\ w_{\eta} w_o - w_{\eta} w_o & \text{if } n_\eta \text{ is odd}. \end{cases}$$

When $n_\eta$ is even, we have

$$\mu_{R(o)}(\eta) = \frac{w_{\eta} w_o}{w_e + w_o} - \frac{w_{\eta} w_o}{w_e + w_o} > 0;$$

and when $n_\eta$ is odd, we have

$$\mu_{R(o)}(\eta) = \frac{w_{\eta} w_o}{w_e + w_o} - \frac{w_{\eta} w_o}{w_e + w_o} < 0.$$ 

Thus in the coupling $\Psi_{R(o)}(R, \sigma, \tau)$, we have

$$\text{Disc}_{\Psi_{R(o)}}(R, \sigma, \tau)(R(e)) = \sum_{\eta: n_\eta \text{ is even}} \mu_{R(o)}(\eta) - \mu_{R(o)}(\eta) = \sum_{\eta: n_\eta \text{ is even}} w_{\eta} (1-\mu^2) \left(\frac{w_e + w_o}{w_e + w_o^2} + w_o\right) = \frac{w_e w_o (1-\mu^2)}{(w_e + w_o)(w_e + w_o^2 + w_o)}.$$ 

We then show that $\lambda \leq \frac{w_o}{w_e^2} \leq \frac{1}{\lambda^2}$. We assume a total order on all edges. For any $\eta$ with even $n_\eta$, let $\phi(\eta)$ be the configuration resulting from flipping the state of $\eta$ on the first edge in $R(e)$. Note that $\phi$ is a bijection between configurations in $\{0, 1\}^R$ with even $n_\eta$ and those with odd $n_\eta$. It is easy to verify that $w_{\eta} \geq \lambda w_{\phi(\eta)}$ if $n_\eta$ is even and $w_{\eta} \geq \mu^2 w_{\phi(\eta)}$ if $n_\eta$ is odd. Combining with the fact that $\phi$ is a bijection, we prove that $\lambda \leq \frac{w_o}{w_e^2} \leq \frac{1}{\lambda^2}$. Substituting this into (6.6), we have $\text{Disc}_{\Psi_{R(o)}}(R, \sigma, \tau)(R(e)) \leq \frac{1-\mu^2}{(1+\lambda\mu^2)^2}$. And since $\mu, \lambda > 0$, all boundary configurations are $R$-feasible, thus for any boundary configurations $\sigma', \tau'$ differing on edges in $R(e)$, we have $T(\sigma', \tau') \leq |R(e)| \leq \Delta$, i.e., we can migrate from one boundary configuration to another by modifying one edge at a time without violating the feasibility. Therefore, if $\Delta < \frac{(1+\lambda\mu^2)^2}{1-\mu^2}$, then

$$\text{Disc}_{\Psi_{R(o)}}(R, \sigma, \tau)(R(e)) \cdot \max_{\sigma_1, \sigma_2 \in \{0, 1\}^{|R(o)|}} T(\sigma_1, \sigma_2) \left(1 - \frac{(1+\lambda\mu^2)^2}{1-\mu^2} \right) \cdot \Delta < 1.$$
For any \( e_0 \in R \), we can apply the recursion in Lemma 6.2 for \( \text{dist}(e, e_0) \) many times where \( \text{dist}(e, e_0) \) denotes the distance between \( e \) and \( e_0 \), thus \( \text{Disc}_q(R, \sigma, \tau) (e_0) = \exp(\Omega(-\text{dist}(e, e_0))) \). Then applying Lemma 6.1, since \( T(\sigma', \tau') \leq \lvert E \rvert \leq n^2 \) for any \( \sigma' \) and \( \tau' \), we have the strong spatial mixing.

The ferromagnetic Ising model is a spin system specified by \( \Phi_E : \{0,1\}^2 \to \mathbb{R}^+ \) and \( \Phi_V : \{0,1\} \to \mathbb{R}^+ \) such that \( \Phi_E(x,y) = a > 1 \) if \( x = y \) and \( \Phi_E(x,y) = 1 \) otherwise; \( \Phi_V(x) = b > 0 \) if \( x = 1 \) and \( \Phi_V(x) = 1 \) if \( x = 0 \). We call \((a,b)\) the parameters of the system.

The Ising model can also be specified by the inverse temperature \( \beta > 0 \) and external field \( B \) as follows. Given a graph \( G = (V,E) \), the partition function is defined as

\[
Z_{\text{Ising}}(G) = \sum_{\sigma \in \{-1,1\}^V} \exp(-\beta H(\sigma)),
\]

where the Hamiltonian \( H(\sigma) \) is given by

\[
H(\sigma) = -\sum_{uv \in E} \sigma(u)\sigma(v) - B \sum_{v \in V} \sigma(v).
\]

**Theorem 6.2. (Jerrum and Sinclair [36])** Let \( G(V,E) \) be a graph. Let \( \lambda = \frac{a-1}{b+1} = \tanh \beta, \mu = \left| \frac{b-1}{b+1} \right| = \tanh \beta B \), then

\[
Z_{\text{Ising}}(G) = M_G \cdot Z_{\text{sub}}(G),
\]

for some \( M_G \) which can be computed in polynomial time.

The transformation from the ferromagnetic Ising model to the subgraphs world model is actually a holomorphic transformation and the above theorem can be seen as a special case of Valiant’s Holant theorem [13,57].

Translating the conditions in Theorem 6.1 for subgraphs world back to the ferromagnetic Ising model, we have that \( \Delta < \frac{(ab^2 + a + 2b)^2}{b(5+1) + (6+1)^2} \) or equivalently \( \Delta < \left( \frac{2ab^2 + 2ab + a + b}{(a + b)(a + b + 1)} \right)^2 \). A simpler sufficient condition is that \( \Delta < \frac{1}{4} \left( \frac{2ab^2 + 2ab + a + b}{a + b} \right)^2 \).

### 6.3 Ferromagnetic Potts Model

Let \( G(V,E) \) be an undirected graph, and \( \Phi : [q]^2 \to \mathbb{R}^+ \) be a symmetric function of nonnegative values. Consider the \( q \)-state spin system whose partition function is defined by

\[
Z(G) = \sum_{\sigma \in [q]^V} w(\sigma)
\]

where \( w(\sigma) = \prod_{(u,v) \in E} \Phi(\sigma(u), \sigma(v)) \).

The Potts model is a \( q \)-state spin system defined as that \( \Phi(x,y) = \lambda \) if \( x = y \) and \( \Phi(x,y) = 1 \) if otherwise. We also write \( \lambda = e^\beta \) where \( \beta \) is the inverse temperature. The weight of a configuration \( \sigma \in [q]^E \) is \( w(\sigma) = \lambda^\text{monoch}(E) \), where \( \text{monoch}(E) \) is the number of monochromatic edges, i.e., edges \( uv \in E \) that \( \sigma(u) = \sigma(v) \). A Potts model is ferromagnetic if \( \beta > 0 \) or equivalently if \( \lambda > 1 \).

We then state some general framework for the strong spatial mixing by recursive coupling on spin systems. Note that although we can represent a spin system as a Holant problem on the edge-vertex incident graph and all rules for Holant problems follow, we still state a version for the original spin systems where vertices are variables and edges are constraints, because sometimes it is more convenient to analyze the correlation decay in this model.

For \( \Lambda \subseteq V \), a configuration \( \sigma \in [q]^\Lambda \) is \( R \)-feasible if there exists a \( \tau \in [q]^V \) consistent with \( \sigma \) over \( \Lambda \) and \( w(\tau) > 0 \). For a feasible \( \sigma \in [q]^\Lambda \), we can accordingly define the Gibbs measure \( \mu^\sigma \) over \([q]^V\) and the marginal distribution \( \mu^\sigma_v \) at vertex \( v \), as well as the strong spatial mixing on spin systems.

For spin systems, a region is a vertex set \( R \subseteq V \). Its edge boundary is \( \delta R = \{uv \in E \mid u \in R, v \notin R\} \). For a boundary configuration \( \sigma \in [q]^\delta R \), we can similarly define the regional weight as

\[
w^\delta_R(\eta) = \prod_{uv \in E(R,R)} \Phi(\eta(u), \eta(v)) \cdot \prod_{e \in \delta R} \Phi(\sigma(e), \eta(u)),
\]

and a boundary configuration \( \sigma \in [q]^\delta R \) is \( R \)-feasible if there exists an \( \eta \in [q]^R \) such that \( w^\delta_R(\eta) > 0 \). Given an \( R \)-feasible boundary configuration \( \sigma \in [q]^\delta R \), we can accordingly define the regional Gibbs measure \( \mu^\delta_R \) over \([q]^R\) as that \( \mu^\delta_R(\eta) = \frac{w^\delta_R(\eta)}{\sum_{\eta' \in [q]^R} w^\delta_R(\eta')} \).

For any \( v \in R \), let \( \mu^\sigma_v \) be the marginal distribution of \( \mu^\sigma \) at vertex \( v \). And for any coupling \( \Psi(R, \sigma, \tau) \) of \( \mu^\sigma \), \( \mu^\tau \), where \( \sigma, \tau \) are \( R \)-feasible boundary configurations, we also define the discrepancy of \( \Psi(R, \sigma, \tau) \) at vertex \( v \) as that

\[
\text{Disc}_{\Psi}(R, \sigma, \tau)(v) = \text{Pr}_{(\eta,\eta') \sim \Psi(R, \sigma, \tau)}[\eta(v) \neq \eta'(v)].
\]

Let \( R \subseteq V \) be some region, \( v_0 \in R \), and \( \sigma, \tau \in [q]^\delta R \) be any two \( R \)-feasible boundary configurations that differ on only one edge \( uv \in \delta R \) where \( u \in R \) and \( v \notin R \). Let \( E_R(v) = \{uv \in E \mid u \in R\} \) be the set of internal edges of \( R \) incident to \( v \).

Let \( \Psi_v(R, \sigma, \tau) \) be a coupling of marginal distributions \( \mu^\sigma_{R,v}, \mu^\tau_{R,v} \) at vertex \( v \). Due to [35], a coupling \( \Psi_v(R, \sigma, \tau) \) of \( \mu^\sigma_{R,v}, \mu^\tau_{R,v} \) can be recursively constructed by the coupling rule \( \Psi_v(R, \sigma, \tau) \) at the vertices \( v \) incident to the only disagreeing edge, by the same routine stated for Holant problems in Section 6.1.
Specific to the ferromagnetic Potts model (or generally all spin systems with soft constraints), we have the following lemma.

**Lemma 6.3.** For spin systems with positive-valued $\Phi$, for any family of graphs whose maximum degree is bounded by $\Delta$. A ferromagnetic Potts model on graph family $G$ has strong spatial mixing if $q - 2 > (\lambda - 1) (\Delta - 1) \lambda^\Delta$, or in terms of inverse temperature if $\beta < \frac{\ln(\frac{\Delta+1}{\Delta})}{\Delta+1}$.

**Proof.** Let $R \subseteq V$ be a region and $\sigma, \tau \in [q]^{\delta R}$ be two $R$-feasible boundary configurations differing on $v = uv \in \delta R$ with $v \in R$ and $u \notin R$. The regional weights $w_R, w_R'$ and regional Gibbs measures $\mu_R, \mu_R'$ can be defined accordingly.

Let $\Psi_v(R, \sigma, \tau)$ be the coupling of $\mu_{R,v}^\sigma, \mu_{R,v}^\tau$ that $P_{\Psi_v(R,\sigma,\tau)}(\eta(v)) = \eta'(v))$ is maximized, i.e. the discrepancy $\text{Disc}_{\Psi_v(R,\sigma,\tau)}(v)$ at $v$ is minimized. We first give an upper bound on $\text{Disc}_{\Psi_v(R,\sigma,\tau)}(v)$.

Consider a boundary configuration $\sigma'$ that agrees with $\sigma$ on $\delta R \setminus \{v\}$. Let $\sigma'(e) = q$, a free color not in $[q]$, and override the definition of function $\Phi$ such that $\Phi(i, q) = \Phi(q, i) = 1$ for all $i \in [q]$. Let $c_1 = \sum_{\eta(e) = 0, \tau(e) = 1} w_R(\eta)$. Without loss of generality, assume $\sigma(e) = 0, \tau(e) = 1$ and $c_0 \geq 1$. Denote that $c = \sum_{i=2}^{q-1} c_i$. We have

$$\sum_{\eta \in [q]^R \atop \eta(v) = i} \mu_R^\sigma(\eta) = \begin{cases} \frac{\lambda c_0}{\lambda c_0 + c_0 + c}, & i = 0, \\ \frac{\lambda c_0 + c_0 + c}{\lambda c_0 + c_0 + c}, & 1 \leq i < q, \end{cases}$$

and

$$\sum_{\eta \in [q]^R \atop \eta(v) = i} \mu_R^\sigma(\eta) = \begin{cases} \frac{\lambda c_0}{\lambda c_0 + c_0 + c}, & i = 1, \\ \frac{\lambda c_0 + c_0 + c}{\lambda c_0 + c_0 + c}, & 0, 2 \leq i < q. \end{cases}$$

For ferromagnetic Potts model, $\lambda > 1$ and $\lambda c_0 + c_1 + c > c_0 + \lambda c_1 + c$, thus $\sum_{\eta \in [q]^R \atop \eta(v) = i} \mu_R^\sigma(\eta) < \sum_{\eta \in [q]^R \atop \eta(v) = i} \mu_R^\tau(\eta)$ for all $i \neq 0$. Therefore in the coupling $\Psi_v(R, \sigma, \tau)$, we have

$$\text{Disc}_{\Psi_v(R,\sigma,\tau)}(v) = \left( \sum_{\eta \in [q]^R \atop \eta(v) = 0} \mu_R^\sigma(\eta) - \sum_{\eta \in [q]^R \atop \eta(v) = 0} \mu_R^\tau(\eta) \right) = \lambda c_0 + c_1 + c - c_0 \leq \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c} = \left( \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c} \right) \leq \left( \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c} \right) = \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c}.$$ 

The value of the last term $\frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c}$ depends on $R, \sigma$ and $c$. For fixed $c$, define $\nu(R, \sigma) = \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c}$, where $c_0, c$ are defined by $R, \sigma$, and $c$ as above.

Let $R' \subseteq R$ be any region containing $v$. Let $\pi \in [q]^{\delta R'}$ be an $R'$-feasible boundary configuration that agrees with $\sigma$ on common edges and maximizes $\nu(R', \pi)$. The next lemma is very similar to the one proved in [34].

**Claim 6.1.** $\nu(R, \sigma) \leq \nu(R', \pi)$.

**Proof.** Let $\eta$ be an $R'$-feasible boundary configuration of $R'$ such that $\eta(e) = q$ the free color and $\eta$ agrees with $\sigma$ on all other common edges. We use $c_i^\eta$ to denote the sum of weight configurations in $R'$ which assign spin $i$ to $v$ with boundary configuration $\eta$ and $c_i^\eta = \sum_{\eta \in [q]^R \atop \eta(v) = i} w_R(\eta)$. Recall that $\nu(R, \sigma) = \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c}$. Then the claim follows from the fact that $c_i/c_0$ is a convex combination of $c_i^\eta/c_0^\eta$ over all such $\eta$.

Consider $R' = \{v\}$, suppose $v$ has $k$ neighbors in $R$ and the vertex boundary of $R$, where $0 \leq k < \Delta$ ($u$ is not taken into account as it is always assigned free color in the definition of $c_i s$). We want to find a boundary condition $\pi$ that maximize $\nu(R', \pi)$, which is equivalent to minimize $\nu(R, \sigma)$.

Since $\lambda > 1$, the ratio achieves its minimum when $\pi$ assigns all the $k$ incident edges of $v$ with spin $0$. In this case, $c_0 = \lambda^k, c = \sum_{i=2}^{q-1} c_i = q - 2$. Thus assuming that $q - 2 > (\lambda - 1) (\Delta - 1) \lambda^\Delta$, we have

$$\text{Disc}_{\Psi_v(R,\sigma,\tau)}(v) \leq \nu(R, \sigma) \leq \nu(R', \pi) \leq \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c} \leq \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c} = \left( \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c} \right) \leq \left( \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c} \right) = \frac{\lambda c_0 + c_1 + c}{\lambda c_0 + c_1 + c}.$$ 

On the other hand $|E_R(v)| \leq \Delta$. Applying Lemma 6.3, we have the strong spatial mixing.
6.4 Algorithmic implications. Both subgraphs world and ferromagnetic Potts model are Holant problems of regular constraint functions and both satisfy tractable search. Then by Theorem 5.3, we have the following algorithmic results.

**Theorem 6.4.** Let \( G \) be the family of apex-minor-free graphs of maximum degree \( \Delta \).

- If \( \Delta < \frac{(1+\mu)^2}{1-\mu^2} \), there exists an FPTAS for subgraphs world of parameters \( 0 < \mu, \lambda < 1 \) on graphs from \( G \).
- If \( \Delta < \frac{1}{4} \left( \frac{e^{2\beta B} + e^{-2\beta B}}{e^{2\beta B} + e^{-2\beta B}} \right)^2 \), there exists an FPTAS for ferromagnetic Ising model of inverse temperature \( \beta \) and external filed \( B \) on graphs from \( G \).
- If \( \beta < \frac{\ln(2+1)}{2\Delta + 1} \), there exists an FPTAS for \( q \)-state ferromagnetic Potts model of inverse temperature \( \beta \) on graphs from \( G \).

The FPTAS for Ising model is not by applying Theorem 5.3 but due to the transformation between subgraphs world and Ising model.

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**References**


